Quantum Physics III (8.06) Spring 2006
Solution Set 4

March 3, 2006

1. **Landau Levels: numerics (4 points)** When $B = 10$ tesla, then the energy spacing $\hbar\omega_L$ is given by

$$\begin{aligned} \hbar\omega_L &= \frac{\hbar c e B}{mc^2} = \frac{(197 \times 10^{-7}\,\mathrm{eVcm})(300 \times 10^5\,\mathrm{eV/cm})}{511\,\mathrm{keV}} \\ &= 1.2 \times 10^{-3}\,\mathrm{eV}. \end{aligned}$$

In natural units the length ℓ_0 is

$$\ell_0 = \sqrt{\frac{\hbar}{m\omega_L}} = \hbar c \sqrt{\frac{1}{mc^2 \hbar\omega_L}} = 4.1 \times 10^{-2} \frac{\hbar c}{\mathrm{eV}}.$$

Notice that this length is the inverse of the geometric average of the two energy scales of the problem: the rest energy of the electron, and the cyclotron energy $\hbar\omega_L$. Finally, in cm, $\ell_0 = 8.0 \times 10^{-7}$cm.

2. **Transformation between basis vectors of different gauges (12 points)**

(a) **(2 points)** Consider the difference

$$\begin{aligned} \vec{A}' - \vec{A} &= (By, Bx, 0) \\ &= -(\partial_x f, \partial_y f, 0). \end{aligned}$$

The solution is easy to guess

$$f(x, y) = -Bxy.$$

(b) **(5 points)** Consider $\langle x_0|$ in $\langle y_0|$ basis.

$$\langle x_0| = \int_{-\infty}^{+\infty} dy_0\ \langle x_0|y_0\rangle\ \langle y_0| \tag{1}$$

To find $\langle x_0|y_0\rangle$ we consider the following equalities

$$\begin{aligned} \langle x_0|Y|y_0\rangle &= y_0\langle x_0|y_0\rangle & (2) \\ &= i l_0^2 \partial_{x_0}\langle x_0|y_0\rangle & (3) \end{aligned}$$

where in the first equality we act Y to the right and in the second equality we act Y to the left. The right hand sides of equations (2) and (3) give rise to a differential equation for $\langle x_0|y_0\rangle$ which can be solved by

$$\langle x_0|y_0\rangle = Ce^{-i\frac{x_0}{l_0^2}y_0}$$

with C a constant of integration. Setting[1] $C = \frac{1}{\sqrt{2\pi l_0^2}}$ we find from (1)

$$\langle x_0| = \frac{1}{\sqrt{2\pi l_0^2}} \int_{-\infty}^{+\infty} e^{-i\frac{x_0}{l_0^2}y_0} \langle y_0|. \tag{4}$$

[1] The normalization constant C can be fixed by requiring $\langle x_0|x_0'\rangle = \delta(x_0 - x_0')$.

Alternatively, one can solve this problem by comparing the commutation relation $[X, Y] = -il_0^2$ to the canonical commutator of momentum p and coordinate x, $[p, x] = -i\hbar$. We know from 8.05 that $[p, x] = -i\hbar$ implies $p = -i\hbar d/dx$, which allows to translate all results of one particle QM in d=1 to the Landau level problem in d=2. This is achieved by replacing $p \to X$, $x \to Y$, $\hbar \to l_0^2$, etc. After this identification, the relation we are seeking to establish in this problem is nothing but the familiar $\langle p| = \sum_x e^{-ipx/\hbar}\langle x|$.

(c) **(5 points)** Consider the wavefunction of lowest energy eigenstate in the unprimed system:

$$\begin{aligned}\psi_0(x, y; y_0) &= \frac{1}{(\pi l_0^2)^{\frac{1}{4}}} e^{-\frac{iy_0 x}{l_0^2}} e^{-\frac{(y-y_0)^2}{2l_0}} \\ &= \frac{l_0}{(\pi l_0^2)^{\frac{1}{4}}} e^{-\frac{iy_0 x}{l_0^2}} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} e^{ik(y-y_0)-\frac{l_0^2 k^2}{2}}\end{aligned} \tag{5}$$

Recall the formula for the delta function

$$\int_{-\infty}^{\infty} e^{ik(x-x_0)} dk = 2\pi\delta(x - x_0) \tag{6}$$

Now we plug (5) into the right hand side of equation (10) of psets 4, exchange the order of integration and use (6) to compute the integrals:

$$\begin{aligned}\text{RHS} &= e^{\frac{ixy}{l_0^2}} \int_{-\infty}^{\infty} \frac{dy_0}{\sqrt{2\pi l_0^2}} e^{\frac{ix_0 y_0}{l_0^2}} \frac{l_0 e^{-\frac{iy_0 x}{l_0^2}}}{(\pi l_0^2)^{\frac{1}{4}}} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} e^{ik(y-y_0)-\frac{l_0^2 k^2}{2}} \\ &= \frac{1}{(\pi l_0)^{\frac{1}{4}}} e^{\frac{ixy}{l_0^2}} \int_{-\infty}^{\infty} e^{iky-\frac{l_0^2 k^2}{2}} dk \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} dy_0 e^{iy_0\left(\frac{x_0-x}{l_0^2}-k\right)} \right) \\ &= \frac{1}{(\pi l_0)^{\frac{1}{4}}} e^{\frac{ixy}{l_0^2}} \int_{-\infty}^{\infty} \delta\left(\frac{x_0 - x}{l_0^2} - k\right) e^{iky-\frac{l_0^2 k^2}{2}} dk \\ &= \frac{1}{(\pi l_0)^{\frac{1}{4}}} e^{\frac{ix_0 y}{l_0^2} - \frac{(x_0-x)^2}{2l_0^2}} \\ &= \text{LHS}.\end{aligned}$$

3. **Landau levels in symmetric gauge (21 points)**

(a) **(3 points)** Given that

$$[v_x, v_y] = \frac{i\hbar\omega_L}{m} \ . \tag{7}$$

the computation for the commutator $[a, a^\dagger]$ is

$$\begin{aligned}[a, a^\dagger] &= \frac{m}{2\hbar\omega_L}[v_x + iv_y, v_x - iv_y] \\ &= -i\frac{m}{\hbar\omega_L}[v_x, v_y] \\ &= -i\frac{m}{\hbar\omega_L} i\frac{\hbar\omega_L}{m} \\ &= 1.\end{aligned}$$

Now we check that $H = \hbar\omega_L(a^\dagger a + \frac{1}{2})$ is indeed the correct Hamiltonian:

$$\begin{aligned}
\hbar\omega_L(a^\dagger a + \frac{1}{2}) &= \hbar\omega_L \left(\frac{m}{2\hbar\omega_L} \left(v_x^2 + v_y^2 + i[v_x, v_y]\right) + \frac{1}{2} \right) \\
&= \frac{m\vec{v}^2}{2}
\end{aligned}$$

where we have used again (7).

(b) **(4 points)** Since $\vec{p} = -i\hbar\frac{\partial}{\partial\vec{x}}$ we can write the the complex derivatives as

$$\begin{aligned}
\partial_z &= \frac{1}{2}(\partial_x - i\partial_y) = \frac{i}{2\hbar}(p_x - ip_y) \\
\partial_{\bar{z}} &= \frac{1}{2}(\partial_x + i\partial_y) = \frac{i}{2\hbar}(p_x + ip_y)
\end{aligned} \tag{8}$$

Also we observe that

$$\frac{z}{2l_0} = \frac{x}{2l_0} + i\frac{y}{2l_0} = \frac{A_y - iA_x}{Bl_0} \tag{9}$$

Plugging (8), (9) this in the right hand side of the desired relation we get

$$\begin{aligned}
a &= \sqrt{\frac{m}{2\hbar\omega_L}}(v_x + iv_y) \\
&= \sqrt{\frac{m}{2\hbar\omega_L}}\frac{1}{m}\left(-\frac{q}{c}(A_x + iA_y) + (p_x + ip_y)\right) \\
&= \frac{l_0}{\sqrt{2}}\left(\frac{-A_x - iA_y}{Bl_0^2} + \frac{1}{\hbar}(p_x + ip_y)\right) \\
&= -\frac{i}{\sqrt{2}}(\frac{z}{2l_0} + 2l_0\partial_{\bar{z}})
\end{aligned}$$

(c) **(3 points)** Plugging the definition of a in the annihilation condition $a|\psi\rangle = 0$ we get

$$\partial_{\bar{z}}\psi = -\frac{z}{4l_0^2}\psi.$$

This can be viewed as an ordinary differential equation for ψ with $\bar{z}$ as independent variable and z as a parameter. The general solution is

$$\psi(z, \bar{z}) = f(z)e^{-\frac{z\bar{z}}{4l_0^2}}$$

where $f(z)$ is an arbitrary function of z.

(d) **(4 points)** The basis of the ground state wave functions is

$$\psi_n = N_n z^n e^{-\frac{z\bar{z}}{4l_0^2}}$$

We compute the normalization of ψ_n by first going to the polar coordinates r, ϕ. Then the integral over ϕ is easily computed. To take the integral over r we perform another change of variables $\rho = r^2$:

$$
\begin{aligned}
1 = \langle\psi_n|\psi_n\rangle &= \frac{N_n^2}{2}\int dz d\bar{z}(z\bar{z})^n e^{-\frac{z\bar{z}}{2l_0^2}} \\
&= N_n^2 \int_0^\infty \int_0^{2\pi} dr d\phi r^{2n+1} e^{-\frac{r^2}{2l_0^2}} \\
&= \pi N_n^2 \int_0^\infty d\rho \rho^n e^{-\frac{\rho}{2l_0^2}} \\
&= \pi N_n^2 (-1)^n \frac{\partial^n}{\partial\alpha^n}\left(\int_0^\infty e^{-\alpha\rho} d\rho\right)\Bigg|_{\alpha=\frac{1}{2l_0^2}} \\
&= \pi N_n^2 (-1)^{n+1} \frac{\partial^n}{\partial\alpha^n}\left(\frac{1}{\alpha}\right)\Bigg|_{\alpha=\frac{1}{2l_0^2}} \\
&= \pi N_n^2 n!(2l_0^2)^{n+1}
\end{aligned}
$$

and therefore

$$N_n = (\pi n! 2^{n+1} l_0^{2n+2})^{-\frac{1}{2}}.$$

Next we compute $\langle\psi_n|x^2+y^2|\psi_n\rangle$ by first noticing that $x^2+y^2 = z\bar{z}$, and second dividing by[2] $\langle\psi_n|\psi_n\rangle$

$$
\begin{aligned}
\langle\psi_n|x^2+y^2|\psi_n\rangle &= \langle\psi_n|z\bar{z}|\psi_n\rangle \\
&= \frac{\langle\psi_n|z\bar{z}|\psi_n\rangle}{\langle\psi_n|\psi_n\rangle} \\
&= -\left[\frac{\partial^{n+1}}{\partial\alpha^{n+1}}\left(\frac{1}{\alpha}\right) \Bigg/ \frac{\partial^n}{\partial\alpha^n}\left(\frac{1}{\alpha}\right)\right]\Bigg|_{\alpha=\frac{1}{2l_0^2}} \\
&= 2(n+1)l_0^2.
\end{aligned}
$$

(e) **(3 points)** The contour plots of the probability densities $|\psi_n(x,y)|^2$ for $n = 0, 4, 10$ are shown on the figures 1, 2 and 3 respectively

(f) **(4 points)** The angular momentum L in complex notations is

$$
\begin{aligned}
L &= xp_y - yp_x \\
&= (-i\hbar)\frac{i}{2}\left((z+\bar{z})(\partial_z - \partial_{\bar{z}}) + (z-\bar{z})(\partial_z + \partial_{\bar{z}})\right) \\
&= \hbar\left(z\partial_z - \bar{z}\partial_{\bar{z}}\right).
\end{aligned}
$$

[2] We do it for convenience, this way we don't need to keep trace of normalization constant, as it cancels out and we can use the insight we got when we computed $\langle\psi_n|\psi_n\rangle$

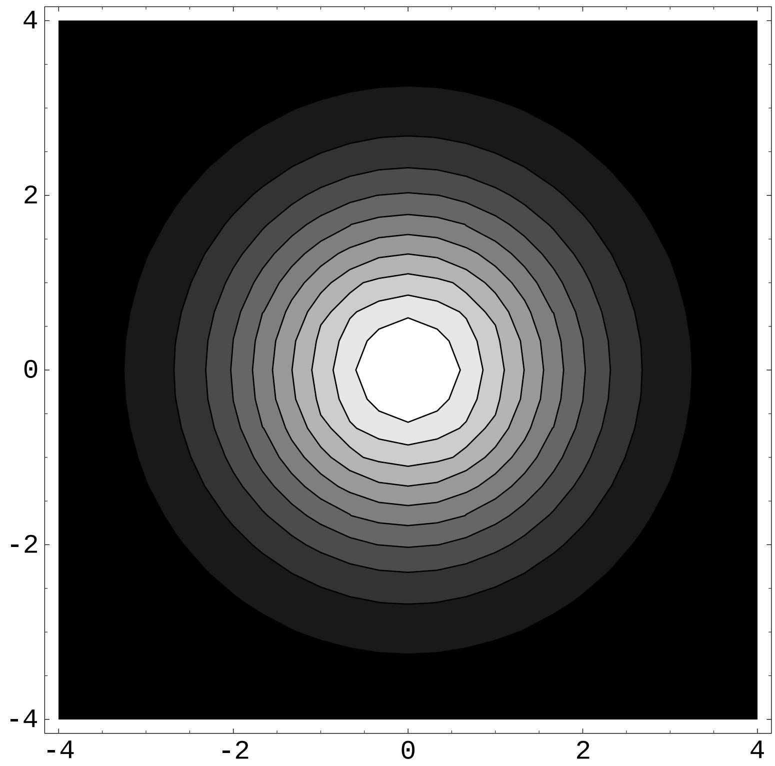

Figure 1: $|\psi_0(x,y)|^2$ is simply a gaussian which has a peak at 0.

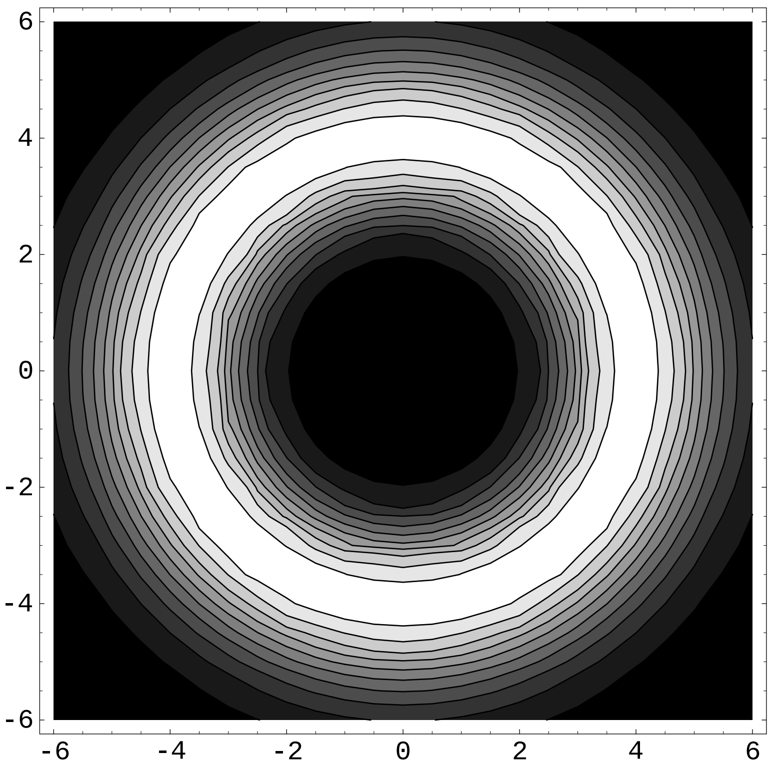

Figure 2: We see, that for $n = 4$ the probability to detect the particle is concentrated in the circular area around th origin

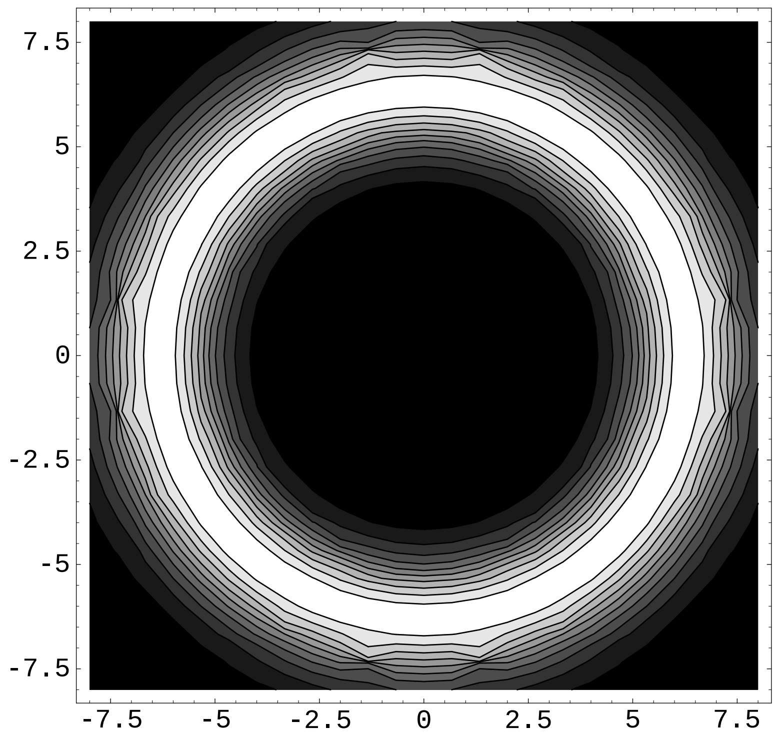

Figure 3: The case of $n = 10$ is qualitatively similar to the $n = 4$ case, the radius of the ring grows roughly linearly with n, which, in agreement with part (d)

Let us compute the commutator of angular momentum L with raising operator $a^\dagger$

$$
\begin{aligned}
[L, a^\dagger] &= -\frac{i\hbar}{\sqrt{2}}\left[z\partial_z - \bar{z}\partial_{\bar{z}}, \frac{z}{2l_0} + 2l_0\partial_{\bar{z}}\right] \\
&= -\frac{i\hbar}{\sqrt{2}}\left(\left[z\partial_z, \frac{z}{2l_0}\right] - [\bar{z}\partial_{\bar{z}}, 2l_0\partial_{\bar{z}}]\right) \\
&= -\frac{i\hbar}{\sqrt{2}}\left(\frac{z}{2l_0} + 2l_0\partial_{\bar{z}}\right) \\
&= \hbar a^\dagger .
\end{aligned}
$$

Similarly

$$[L, a] = -a.$$

The commutator of angular momentum with the Hamiltonian is

$$
\begin{aligned}
[L, H] =& [L, \hbar\omega_L(a^\dagger a + \frac{1}{2})] \\
&= \hbar\omega_L\left([L, a^\dagger]a + a^\dagger[L, a]\right) \\
&= 0.
\end{aligned}
$$

Acting with L on ψ_n we obtain

$$
\begin{aligned}
L\psi_n &= \hbar N_n(z\partial_z - \bar{z}\partial_{\bar{z}})z^n e^{-\frac{z\bar{z}}{4l_0^2}} \\
&= \hbar N_n n z_n e^{-\frac{z\bar{z}}{4l_0^2}} \\
&= \hbar n\psi_n .
\end{aligned}
$$

4. **Counting States in the lowest Landau Level in the symmetric gauge (6 points)**

(a) **(3 points)** The probability density is

$$
\begin{aligned}
P &= 2\pi r|\psi_n^2|dr \\
&= 2\pi N_n^2 r^{2n+1} e^{-\frac{r^2}{2l_0^2}}
\end{aligned}
$$

Taking derivative with respect to r and setting it to zero we get

$$(2n+1) - \frac{r_{\max}^2}{l_0^2} = 0.$$

Therefore

$$r_{\max} = l_0\sqrt{2n+1}$$

(b) **(3 points)** The largest value of $n_{\max}$ allowed by the size of the material is determined from

$$l_0\sqrt{2n_{\max}+1} \approx R.$$

Therefore

$$n_{\max} \approx \frac{1}{2}\frac{R^2}{l_0^2} = \frac{\pi R^2 qB}{hc} = \frac{\Phi}{\Phi_0}.$$

5. **Coherent state in the symmetric gauge (12 points)**

Note: In this problem we call the eigenstate of a as ϕ_0 in order to avoid confusion with the ground state eigenfunctions ψ_n defined in problem 3.

(a) **(3 points)** The eigenstate equation

$$a\phi_0 = -i\frac{z_0}{\sqrt{2}l_0}\phi_0 \tag{10}$$

simplifies to

$$\partial_{\bar{z}}\phi_0 = -\frac{z-z_0}{4l_0^2}\phi_0.$$

As we did in problem 3 we view this as an ordinary differential equation for $\phi_0(\bar{z})$. The general solution is

$$\phi_0(z,\bar{z}) = f(z)e^{-\frac{(z-z_0)(\bar{z}-\bar{z}_0)}{4l_0^2}} \tag{11}$$

where $f(z)$ is an arbitrary function of z. *Here and below, one gets full credit for any choice of $f(z)$.*

(b) **(3 points)** Multiplying the wave function by it's complex conjugate we have

$$\begin{aligned}|\phi_0|^2 &= |f(z)|^2 e^{-\frac{(z-z_0)(\bar{z}-\bar{z}_0)}{2l_0^2}} \\ &= |f(z)|^2 e^{-\frac{(x-x_0)^2+(y-y_0)^2}{2l_0^2}}.\end{aligned}$$

(c) **(5 points)** *This part count as extra bonus credit.*

Since the Hamiltonian is time-independent, the wave function at time t is given by

$$\phi(t) = e^{-iHt}\phi_0.$$

In order to find $\phi(t)$ we act with e^{-iHt} on both sides of (10)

$$e^{-iHt}ae^{iHt}e^{-iHt}\phi_0 = -i\frac{z_0}{\sqrt{2}l_0}e^{-iHt}\phi_0$$

which leads to

$$a(-t)\phi(t) = -i\frac{z_0}{\sqrt{2}l_0}\phi(t) \tag{12}$$

where $a(t)$ is the Heisenberg operator for a

$$a(t) = e^{iHt}ae^{-iHt} .$$

Since $a(t)$ satisfies the equation

$$\frac{da}{dt} = -\frac{i}{\hbar}[a,H] = -i\omega_L$$

we find that

$$a(-t) = e^{i\omega_L t}a . \tag{13}$$

Plugging (13) into (12) we get

$$
\begin{aligned}
a\phi(t) &= -i\frac{z_0 e^{-i\omega_L t}}{\sqrt{2}l_0}\phi(t) \\
&= -i\frac{z_0(t)}{\sqrt{2}l_0}\phi(t)
\end{aligned}
$$

with

$$z_0(t) = e^{-i\omega_L t}z_0$$

$\phi(t)$ satisfies exactly the same equation as ϕ_0 with z_0 replaced by $z_0(t)$ (compare with equation (10)). Thus from (11)

$$\phi(t) = f(z)\exp\left(-\frac{(z - z_0(t))(\bar{z} - \bar{z}_0(t))}{4l_0^2}\right). \tag{14}$$

(d) **(1 point)** Multiplying the answer to (c) by its complex conjugate we obtain

$$|\phi(t)|^2 = |f(z)|^2 e^{-\frac{(x-x_0(t))^2+(y-y_0(t))^2}{2l_0^2}}$$

6. **Off-diagonal conductance in two dimensions (10 points)**

(a) **(3 points)** Using the usual formula for the inverse of a 2×2 matrix, we write

$$\sigma = \frac{1}{\rho_0^2 + \rho_H^2}\begin{pmatrix} \rho_0 & \rho_H \\ -\rho_H & \rho_0 \end{pmatrix},$$

which gives us

$$\sigma_0 = \frac{\rho_0}{\rho_0^2 + \rho_H^2} \qquad\qquad \sigma_H = \frac{\rho_H}{\rho_0^2 + \rho_H^2}.$$

(b) **(3 points)** If ρ_H is non vanishing, then

$$\lim_{\rho_0 \to 0} \sigma_0 = \lim_{\rho_0 \to 0} \frac{\rho_0}{\rho_0^2 + \rho_H^2} = 0.$$

Thus we can have both σ_0 and ρ_0 equal to 0 in the presence of a non vanishing B field.

(c) **(4 points)** In two dimensions, the current density is $j = I/\ell$, that is, current per unit length (rather than current per unit area, as in a three-dimensional sample). Therefore, the equation defining the Hall resistance, $V_y = R_H I_x$, is the same information as $\vec{E} = \rho\vec{j}$. Notice that $E_y = V_y/L$ and $j_x = I_x/L$. Therefore:

$$\rho_H = \frac{E_y}{j_x} = \frac{E_y L}{j_x L} = \frac{V_y}{I_x} = R_H.$$

Thus the Hall resistance of the sample does not depend on its dimensions. Alternatively, we can give a more generic treatment (NOT required for full credit):

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \rho_0 & -\rho_H \\ \rho_H & \rho_0 \end{pmatrix}\begin{pmatrix} j_x \\ j_y \end{pmatrix},$$

therefore, we can write,

$$\begin{aligned}
\begin{pmatrix} V_x \\ V_y \end{pmatrix} &= \begin{pmatrix} W & 0 \\ 0 & L \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} \\
&= \begin{pmatrix} \rho_0 W j_x - \rho_H W j_y \\ \rho_H L j_x + \rho_0 L j_y \end{pmatrix} \\
&= \begin{pmatrix} \rho_0 \frac{W}{L} & -\rho_H \\ \rho_H & \rho_0 \frac{L}{W} \end{pmatrix} \begin{pmatrix} I_x \\ I_y \end{pmatrix},
\end{aligned}$$

which is the matrix equation of the form $V = RI$. Clearly, when $\rho_0 = 0$, the resistance matrix is independent of L/W.