§9. Charged particles in a magnetic field

QM of charged particles in a constant magnetic field gives rise to <u>many fascinating pheomena</u>, including:

Landau levels

integer quantum Hall effect
(fractional)

De Hass-Van Alphen effect

Aharonov - Bohm effect

Condensed matter physics of electrons in materials at low T and high B is a vast area of contemporary experimental and theoretical physics, which we will briefly touch upon.

9.1 ~~Hamiltonian~~ ~~Classical Mechanics of charged particles in a magnetic field~~

Canonical quantization

Classical equation of motion of a charged particle in an EM field

$$m\frac{d^2\vec{x}}{dt^2} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right) \quad (1)$$

write

$$\vec{v} = \frac{d\vec{x}}{dt}$$

How to describe the motion ~~particle~~ in QM?

previously: $$H = -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{x}) \quad (2)$$

However a potential $V(\vec{x})$ can never give rise to velocity-dependent force as in (1). Thus (2) cannot describe a quantum version of (1).

To write down the Schrodinger equation for (1) we need to use a procedure called canonical quantization.

9.1.1 Classical Hamitonian ~~An alternative formulation of Classical mechanics.~~

Instead of directly writing down Newton's equations ~~eeeee~~, one introduces a "Classical Hamiltonian"

$$\mathcal{H}(x_i, p_i), \quad i=1,2,3$$

canonical coordinates — canonical momentum

~~The equations of Newton are then given by the~~ H is chosen so that the so-called Hamitonian equations:

$$\dot{x}_i = \frac{\partial \mathcal{H}}{\partial p_i} \qquad (3) \qquad \boxed{\text{Copy}}$$

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial x_i}$$

The ~~set of 1st order equations (*) can be shown~~ are ~~to be~~ ~~equal~~ equivalent to Newton's equations

$$\vec{F} = m \frac{d^2\vec{x}}{dt^2}$$

~~Remarks~~

① when the system does not depend on time explicitly, $\mathcal{H}$ coincides with total energy

For example: a particle in a potential $V(\vec{x})$

$$\mathcal{H} = \underbrace{\frac{\vec{p}^2}{2m}}_{\text{kinetic energy}} + \underbrace{V(\vec{x})}_{\text{potential energy}}$$

$$(*) \Rightarrow \quad \dot{\vec{x}} = \frac{\vec{p}}{m}, \qquad \dot{\vec{p}} = -\nabla V \quad \Rightarrow \quad m\ddot{\vec{x}} = -\nabla V$$

10 min

• ~~Classical Hamiltonian~~ a charged particle in an EM field.

We need to find a $\mathcal{H}(\vec{x}, \vec{p})$ whose Hamiltonian equations are equivalent to (1). (SN)

Claim:

$$\mathcal{H} = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + q\phi \qquad (4)$$

with $\vec{A}$: vector potential

ϕ: scalar potential

and

ask students

$$\vec{B} = \nabla \times \vec{A}$$

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\phi \qquad (5)$$

Copy

is the Hamiltonian for (1)

(4) can be derived from 1st principle, here we will outline a sketch check that it does give rise to (1): (SN)

1st Hamiltonian equation:

$$\dot{x}_i = \frac{1}{m}\left(P_i - \frac{q}{c}A_i\right) \qquad (6)$$

or

$$P_i = m\dot{x}_i + \frac{q}{c}A_i \qquad (7)$$

2nd Hamiltonian equation:

$$\frac{dP_i}{dt} = \sum_j \frac{q}{mc}\left(P_j - \frac{q}{c}A_j\right)\frac{\partial A_j}{\partial x_i} - q\frac{\partial \phi}{\partial x_i} \qquad (8)$$

Insert (7) into (8): $\Rightarrow$ (1)

$$m\ddot{x}_i + \frac{q}{c}\underbrace{\left(\frac{\partial A_i}{\partial t} + \frac{\partial A_i}{\partial x_j}\dot{x}_j\right)}_{} = \frac{q}{mc}\cdot m v_j \frac{\partial A_j}{\partial x_i} - q\frac{\partial \phi}{\partial x_i}$$

$$\left(\frac{\partial A_i}{\partial t} + \frac{\partial A_i}{\partial x_j}\dot{x}_j\right) = \frac{dA_i}{dt}$$

$$\Rightarrow \quad m\ddot{x}_i = \frac{q}{c}\underbrace{\left(v_j\frac{\partial A_j}{\partial x_i} - v_j\frac{\partial A_i}{\partial x_j}\right)}_{=(\vec{v}\times\vec{B})_i} + q\underbrace{\left(-\frac{1}{c}\frac{\partial A_i}{\partial t} - \frac{\partial \phi}{\partial x_i}\right)}_{=E_i}$$

$$\vec{v}\times\vec{B} = \vec{v}\times(\nabla\times\vec{A}) = \varepsilon_{ijk}v_j\varepsilon_{klm}\partial_l A_m$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})v_j\partial_l A_m = v_j\partial_i A_j - v_j\partial_j A_i \quad \text{QED}$$

15m

Remarks:

1. $\vec{p} = \underline{\underline{m\vec{v} + \frac{q}{c}\vec{A} \neq m\vec{v}}}$ (important in QM)

$\vec{p}$: canonical momentum

$m\vec{v}$: kinetic momentum

2. $\mathcal{H} = \frac{1}{2} m\vec{v}^2 + e\phi$ ($\vec{B}$ does not do work)

$= KE + PE = \text{total energy}$

$\neq \frac{\vec{p}^2}{2m} + e\phi$

3. Gauge invariance. $\vec{E}, \vec{B}$ do not specify $\vec{A}, \phi$ uniquely.

$\vec{A}' = \vec{A} - \nabla f$

$\phi' = \phi + \frac{1}{c}\frac{\partial f}{\partial t}$

(9)

describe the same $(\vec{E}, \vec{B})$ for ANY $f(x,t)$.

Outline

Lec. 5.

Last Lecture:

$$\mathcal{H} = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + q\phi$$

$$\xrightarrow{\text{Hamiltonian equations}} \quad m\frac{d^2\vec{x}}{dt^2} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$$

1) $m\vec{v} = \vec{p} - \frac{q}{c}\vec{A} \neq \vec{p}$

2) $\mathcal{H}$ depends on $(\vec{A}, \phi)$ instead of $(\vec{E}, \vec{B})$ directly

3) Classical theory is gauge invariant

$$(\vec{E}, \vec{B}) \longrightarrow \begin{array}{c}(\vec{A}, \phi) \\ (\vec{A}', \phi')\end{array} \Big\} \text{ related by a gauge transformation}$$

$\nLeftarrow$

(9). are called gauge transformations.

Classical e.o.m. depend only on $(\vec{E}, \vec{B})$

(more fundamental)

End of lecture 4 $^{15 \text{ min}}$

9.1.3. Canonical quantization. (total 20 m below?)

Among many advantages of the Hamiltonian formulation, one important aspect is its close relation with QM. while Newton's equations of motion capture all the classical physics, its relation to QM is less direct than the Hamiltonian formalism.

procedure of obtaining QM H from classical $\mathcal{H}$:

(canonical quantization)

promote p_i, x_i, $\mathcal{H}$ to operators:

$$x_i \longrightarrow \hat{x}_i$$

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$$

$$p_i \longrightarrow \hat{p}_i = -i\hbar\frac{\partial}{\partial x_i}$$

$$\mathcal{H}(p_i, x_i) \longrightarrow H(\hat{p}_i, \hat{x}_i) = H(-i\hbar\frac{\partial}{\partial x_i}, x_i)$$

Schrodinger equation:

$$-i\hbar\frac{\partial}{\partial t}\psi(\vec{x}, t) = H(\hat{p}, \hat{x})\,\psi(x, t)$$

Classical e.o.m $\longrightarrow$ Heisenberg equations for
(Hamiltonian equations) operators

(see Jaffe's notes)

$\frac{1}{2}$

Remarks:

1. canonical quantization: is not a derivation of QM, rather it should be considered as a connection between a QM system and its classical limit, if such a limit exists. The procedure should be tested by ~~expe~~ experiments.

~~e.g. there are many quantum mechanical concepts with no classical analogues, like spin ... then C.Q. does not apply.~~

2. Operator ordering ambiguities.

In going from $\mathcal{H}(x_i, p_i) \longrightarrow H(\hat{x}_i, \hat{p}_i)$

commute — do not commute.

Say if $\mathcal{H}(\hat{x}, p)$ contains a term

$$x^2 p^2$$

in going to QM, is it

$$\hat{x}^2 \hat{p}^2, \quad \hat{x} \hat{p}^2 \hat{x}, \quad \text{or} \quad \hat{p}^2 \hat{x}^2 \ldots$$

or linear combinations of them?

use symmetry principles or ultimately by experiments.

10 min

$\frac{3}{4}$

9.1.4. Schrodinger equation for a particle in EM field and gauge invariance.

$$\mathcal{H} = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + q\phi$$

$\Rightarrow$

$$H = \frac{1}{2m}\left(-i\hbar\nabla - \frac{q}{c}\vec{A}(x,t)\right)^2 + q\phi(x,t)$$

sch. eq.

$$-i\hbar\frac{\partial}{\partial t}\psi(x,t) = \frac{1}{2m}\left(-i\hbar\nabla - \frac{q}{c}\vec{A}(x,t)\right)^2\psi + q\phi\psi \quad (9)$$

Classical e.o.m.: only depend on $\vec{E}, \vec{B}$

QM: sch. eq. ~~does~~ can only be expressed in $\vec{A}$ and ϕ 1

However in nature we only observe $\vec{E}$ and $\vec{B}$, which do not determine $\vec{A}, \phi$ uniquely.

Suppose (A, ϕ)

$\swarrow$ (A', ϕ') $\Rightarrow$ $(\vec{E}, \vec{B})$

related by a gauge transformation $f(x,t)$

do they describe the same QM ?

Answer : yes. due to a special property of

(9).

In pset 3, you will show that :

If $\psi(x,t)$ satisfies (9)

then $$\boxed{\psi'(x,t) = e^{\frac{-iq}{\hbar c} f(x,t)} \psi(x,t)}$$

Solves the sch. eq. with (A', ϕ').

Note:

1) $|\psi'(\vec{x},t)| = |\psi(\vec{x},t)|$

Probabilities are gauge invariant

2) Any physical observables should also be gauge invariant.

In pset 3, you will check that

$\langle\psi|\hat{x}|\psi\rangle$ is

while $\langle\psi|\hat{p}|\psi\rangle$ is not

and $\langle\psi|m\hat{v}|\psi\rangle$ is.

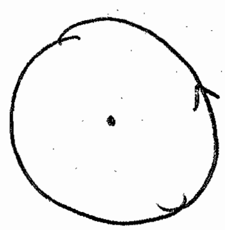
9.2 Charged particles in a constant magnetic field:

9.2.1 Landau levels

consider $\vec{E} = 0$, $\vec{B} = B\hat{e}_z = (0, 0, B)$

classically: circular motion with angular velocity

$$\boxed{\omega_L = \frac{qB}{mc}}$$ Lamor frequency



$$T = \frac{2\pi}{\omega_L}$$

what happens at Quantum level?

To write down the Schrodinger equation first choose a gauge. ($\underline{B\vec{e}_z = \nabla \times \vec{A}}$, $\phi = 0$)

$= \underline{(-\partial_y A_x + \partial_x A_y)}\, \vec{e}_z$

(side)

$+ \underline{\text{cyclic permutations}}$

Some simplest choices:

$$\vec{A} = (0,\ BX,\ 0)$$

or

$$\vec{A}' = (-By,\ 0,\ 0)$$ ✓ (Landau gauge)

or

$$\vec{A}'' = \left(-\tfrac{1}{2}By,\ \tfrac{1}{2}BX,\ 0\right)$$ (symmetric gauge)

Criteria(s): for choosing the gauge:

1) Simplify the problem

2) make important physics manifest.

In the Landau gauge, the Hamiltonian becomes:

$$\hat{H} = \frac{\hat{P}_y^2}{2m} + \frac{1}{2m}\left(\hat{P}_x + \frac{qB}{c}y\right)^2 + \underbrace{\frac{P_z^2}{2m}}_{}\qquad 2\text{(B)}$$

↑ ignore it from now on.

Schrodinger equation:

$$\hat{H}\,\psi(x,y) = E\,\psi(x,y)$$

Notice: $[\hat{H}, \hat{P}_x] = 0$

Can diagonalize $\hat{H}$ and $\hat{P}_x$ together

Let $\psi(x,y) = f(y)\, e^{ik_x x}$ ①

$$\hat{p}_x \psi(x,y) = -i\hbar\, \partial_x \psi(x,y) = \hbar k_x\, \psi(x,y)$$

On ①, $\hat{H}$ becomes:

$$H = \frac{\hat{p}_y^2}{2m} + \frac{1}{2m}\left(\hbar k_x + \frac{qB}{c} y\right)^2$$

c-number

$$= \frac{\hat{p}_y^2}{2m} + \underbrace{\frac{1}{2m}\left(\frac{qB}{c}\right)^2}_{= \frac{m}{2}\left(\frac{qB}{cm}\right)^2}\left(y + \frac{\hbar c k_x}{qB}\right)^2$$

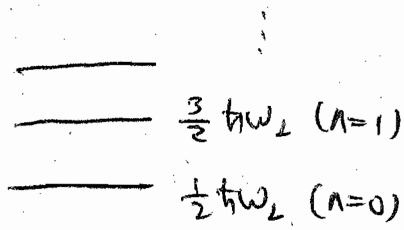
$$= \frac{\hat{p}_y^2}{2m} + \frac{1}{2} m \omega_L^2 (y - y_0)^2 \qquad ②$$

$$\boxed{\omega_L = \frac{qB}{mc}, \quad y_0 = -\frac{\hbar c k_x}{qB}} = -\frac{c p_x}{qB}$$

Classical Lamor frequency

② is S.H.O. with frequency ω_L and shifted

origin $y = y_0$!



$\Rightarrow$

$$E_n = \hbar\omega_L \left(n + \frac{1}{2} \right)$$

$$\psi_{nk_x}(x,y) = e^{ik_x \cdot x} \phi_n(y - y_0)$$

$\uparrow$

w.f. of S.H.O at n-th level.

e.g. $$\phi_0(y-y_0) = \left(\frac{m\omega_L}{\pi\hbar} \right)^{\frac{1}{4}} e^{-\frac{1}{2}\frac{m\omega_L}{\hbar}(y-y_0)^2} \qquad ③$$

Remarks: 1) Energy does not depend on k_x. Any

given n: infinite degeneracy !

n=0: Lowest Landau levels (LLL)

n=1: next LLL etc

2) wave functions:

x-direction: plane wave with momentum $\hbar k_x$

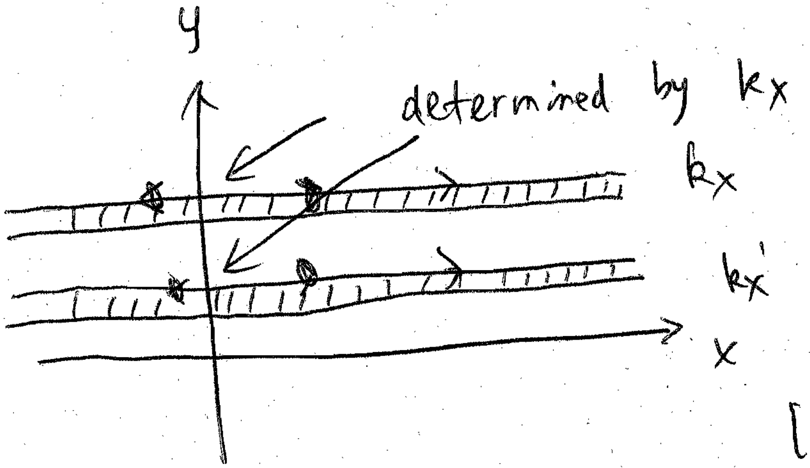
(delocalized)

y-direction: localized around

$$y_0 = -\frac{c p_x}{2B} \propto p_x$$

with a spread controlled by (see 3)

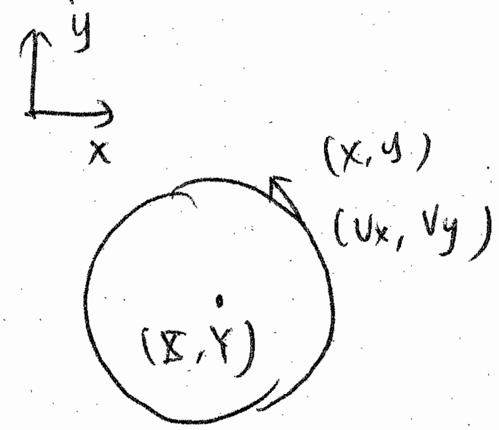
$$\boxed{l_0 = \sqrt{\frac{\hbar}{m\omega_L}} = \sqrt{\frac{\hbar c}{2B}}}$$



$[|k_x| > |k_x'|$

3.) Relation with classical motion :

Classical motion : Circle with center at



$$X = x + \frac{1}{\omega_L} v_y$$

$$Y = y - \frac{1}{\omega_L} v_x$$

(X, Y) conserved numbers.

Projection to y - axis : harmonic oscillation frequency ω_L

centered at Y.

x - axis — — —

centered at X

Landau gauge :

$$X = x + \frac{1}{m\omega_L} \left(p_y - \frac{q}{c} A_y \right) = x + \frac{1}{m\omega_L} p_y$$

$$Y = y - \frac{1}{m\omega_L} \left(p_x - \frac{q}{c} A_x \right) = y - \frac{1}{m\omega_L} \left(p_x + \frac{qB y}{c} \right)$$

$$= - \frac{p_x}{m\omega_L} = - \underline{\underline{\frac{c p_x}{qB}}} = y_0 \ !$$

Classical | Quantum

y: h.o. around y_0 | h.o. around $y_0 = \frac{-cp_x}{2\beta}$

eigenstates of p_x $\downarrow$

x: h.o. around $x + \frac{1}{m\omega_c} p_y$ | plane wave

$\Rightarrow$ Quantum level. x, y motions do not commute!

You will further explore this in Pset 3. (Prob 2)

4) A different gauge:

Suppose we choose $\vec{A} = (0, Bx, 0)$

take previous results. $x \leftrightarrow y$

$B \to -B$

$\Rightarrow$ plane wave in y

S. H. O. in x-direction (Projection of classical motion to x-axis)

Q: $(0, Bx, 0)$ $\qquad$ $(-By, 0, 0)$

plane wave in y $\qquad$ S. H. O. in y

S. H. O in x $\qquad$ plane in x

How can it be ?

<u>Key</u>: the Landau levels are <u>infinite degenerate</u>.

change of gauge $\longrightarrow$ <u>change of basis</u>.

see pset 3.

In pset 4, you will explore the behavior of wave functions in yet another gauge

$$\left(-\tfrac{1}{2}By,\ \tfrac{1}{2}Bx,\ 0\right)$$

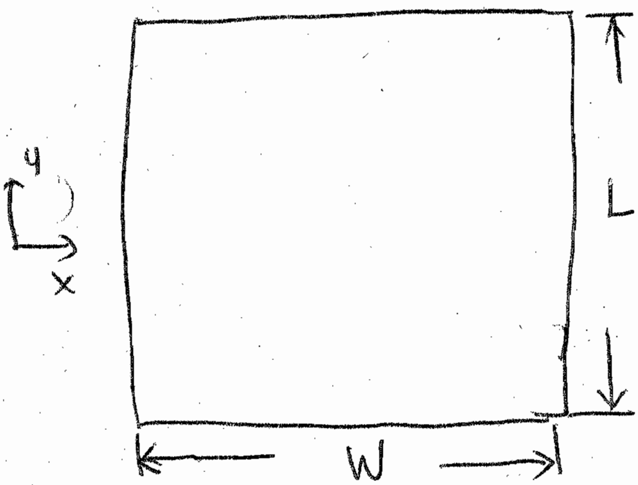
which gauge to choose: depending on specific physical question. in $\overset{F}{\underset{(I)}{Q}}$QHE symmetric gauge is used most.

<u>end of lecture 5</u>

9.2.2 Degeneracy of Landau levels.

so far: a particle in an infinite 2-plane

reality: any sample has a finite size



• set-up

1) use Landau gauge

$(-By, 0, 0)$

2) assume $L, W \gg \ell_0$ = width of wave functions

$\ell_0 = \sqrt{\frac{\hbar c}{eB}}$

3) ~~ignore the boundary effects~~ boundary effect should not be important.
Impose periodic boundary condition in x-direction

(Boundary conditions should not change bulk properties)

Recall Landau gauge wave functions:

$$\psi_n(x, y; k_x) = e^{ik_x x}\,\phi_n(y - y_0), \quad y_0 = \frac{-\hbar c k_x}{eB}$$

- Finite size effects:

1) periodic b.c. in x

$\Rightarrow \quad k_x = \frac{2\pi n}{W}, \quad n \in \mathbb{Z} \quad \Rightarrow \quad \underline{y_0 \text{ quantized}}$

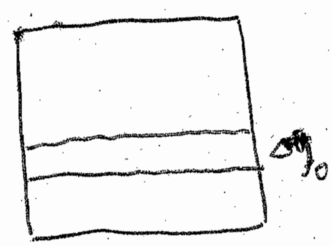
2) y_0 should lie inside the sample $\Rightarrow$ further constrain allowed k_x

spacing ~~between~~ between k_x : $\frac{2\pi}{W}$

spacing ~~b~~ in y_0 : $\boxed{\Delta y_0 = \frac{\hbar c}{eB}\frac{2\pi}{W}}$

$\Rightarrow$ ~~#~~ total # of k_x allowed:

$$\boxed{D = \frac{L}{\Delta y_0} = \frac{WL}{2\pi}\frac{eB}{\hbar c}}$$



(degeneracy of each L.L.)

since $WL = A$

$$\frac{\hbar c}{qB} = \ell_0^2$$

$$\Rightarrow \boxed{D = \frac{WL}{2\pi \ell_0^2} = \frac{A}{A_B}}$$

Note :

1) A_B can be considered as the area occupied by an individual state

$$A_B = 2\pi \ell_0^2 = \frac{hc}{qB} = \underline{\underline{W \Delta y_0}} \quad (\text{picture})$$

2) Flux through A_B is $\Phi_0 = \frac{hc}{q}$

for electrons $\boxed{\Phi_0 = \frac{hc}{e}}$ ← fundamental flux quantum

$$\boxed{D = \frac{\Phi}{\Phi_0} = \frac{A \cdot B}{\Phi_0}}$$ (used later)

9.3 de Haas – Van Alphen effect:

reality: sample has finite size

electron gas. (instead of one electron)

pauli principle

This lecture: dHvA effect

next lecture: Integer quantum Hall effect.

9.3.1 Magnetization

~~where we~~ apply a magnetic field $\vec{B}_0$ to materials.

$\Rightarrow$

Induce magnetic moments $\vec{\mu}_I$

If magnetic field $(\vec{B}_I)$ from $\vec{\mu}_I$

in the same direction as $\vec{B}$ $\Rightarrow$ enhance it

(paramagnetism)

in the opposite direction as $\vec{B}$ $\Rightarrow$ reduce it

(diamagnetism)

Microscopically: ~~we have the following two~~

~~physical mechanisms.~~

Paramagnetism: (due to spin)

Electrons: $\vec{S} \longrightarrow \vec{\mu}_s$

$\vec{B}=0$, $\vec{\mu}_s$ random oriented

$$\vec{E} = -\vec{\mu}_s \cdot \vec{B}$$

apply $\vec{B}$, μ_s will align with $\vec{B}$: enhance it.

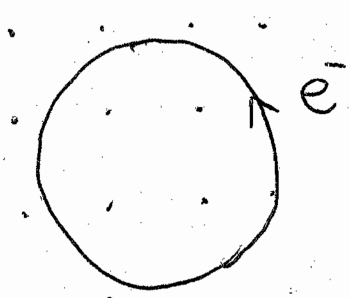
← our focus

Diamagnetism: (due to orbital motion)

① $\vec{B} = B\vec{e}_z$

induced magnetic field: $-\hat{e}_z$

opposite to applied $\vec{B}$-field.



induced magnetic moment:

$$\mu_D = -\frac{\partial E}{\partial B}$$

Ex Example: Single electron in the n-th Landau Level.

$$E = (n+\tfrac{1}{2})\hbar\omega_L = (n+\tfrac{1}{2})\frac{\hbar e}{m_e c} B$$

$$= (2n+1)\mu_B B$$

$$\mu_B = \frac{e\hbar}{2m_e c} \quad : \quad \text{Bohr magneton}$$

$$\boxed{\tfrac{1}{2}\hbar\omega_L = \mu_B B} \quad (\text{on the side})$$

$$\Rightarrow \quad \mu_D = -\frac{\partial E}{\partial B} = -(2n+1)\mu_B$$

Q:

Q: 1) Given a material of area A containing total N electrons, what is

$$M = -\frac{1}{A}\frac{dE_{tot}}{dB}$$

i.e. total induced magnetic moment per unit area.

2) How does M change with B ?

(give DHasV effect)

Idealization: treating electrons as free.
(ignore interaction with ions and among themselves)

9.3.2 Ground States of N electrons in a B-field.

- wave function (Fermi surface)

$\frac{5}{2}\hbar\omega$ $(n=2)$

$\frac{3}{2}\hbar\omega$ $(n=1)$

$\frac{1}{2}\hbar\omega$ $(n=0)$

each level can hold

$$D = \frac{BA}{\Phi_0} \propto B \quad \left(\Phi_0 = \frac{hc}{e} \right)$$

will ignore spin : magnetic field breaks degeneracy between $\uparrow$ and $\downarrow$

treat them separately.

We thus introduce ν (filling ratio):

$$\nu = \frac{N}{D} = \frac{N\Phi_0}{AB}$$

~~In practice: N, A fixed, varying B~~

$0 < \nu < 1$: $n=0$ (LLL) partially filled

$\nu = 1$: LLL ($n=0$) fully filled

$j \leq \nu < j+1$: $n = 0, \cdots j-1$ levels filled

$n = j$ -th level partially filled

In practice: N, A fixed, B varied

convenient to write $\boxed{\nu = \frac{B_0}{B}}$ $\left(\boxed{B_0 = \frac{N\Phi_0}{A}}\right)$ $= n\Phi_0$ electron density

(characterized by material)

$\nu \downarrow \quad B \uparrow$: $\nu < 1 \quad B > B_0$

- Energy of the ground state $\quad (j \le \nu < j+1)$

$$E = D \sum_{n=0}^{j-1} (n+\tfrac{1}{2})\hbar\omega_L + (N - jD)(j+\tfrac{1}{2})\hbar\omega_L$$

Express E in terms of ν (convenient)

$$E = N \underbrace{\frac{\hbar\omega_L}{2}} \left[\frac{1}{\nu} \sum_{n=0}^{j-1} (2n+1) + \left(1 - \frac{j}{\nu}\right)(2j+1) \right]$$

$$= N \underbrace{\mu_B B} \left[(2j+1) - \frac{1}{\nu} j(j+1) \right] \quad \left(\sum_{n=0}^{j-1} 2n+1 = j^2 \right)$$

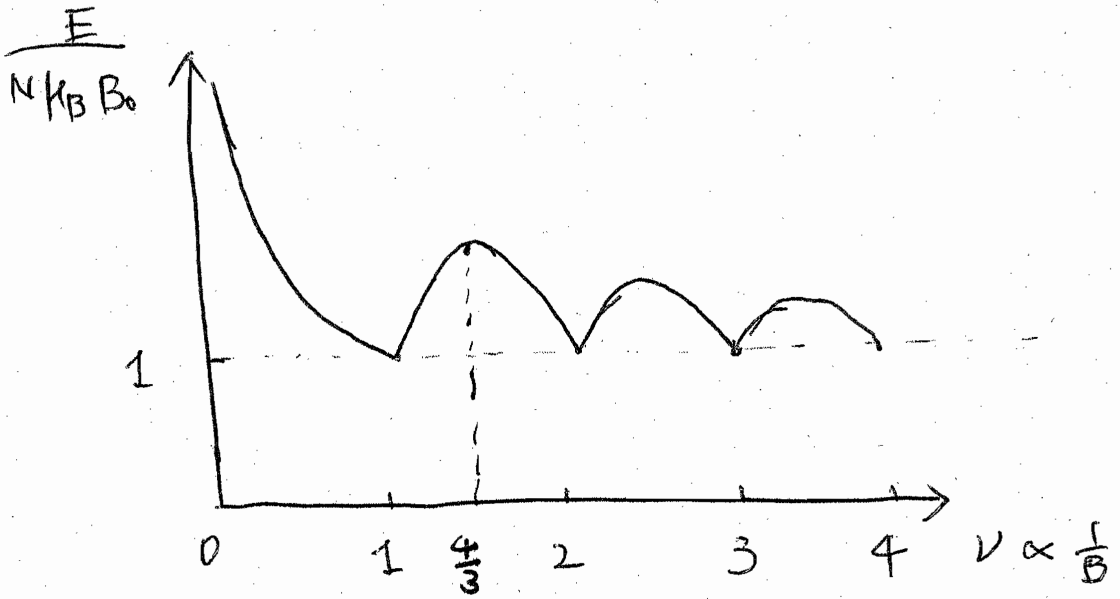
$$= N \mu_B B_0 \left[(2j+1)\frac{1}{\nu} - \frac{1}{\nu^2} j(j+1) \right]$$

$$= N \mu_B B_0 \left[(2j+1)\frac{B}{B_0} - j(j+1)\frac{B^2}{B_0^2} \right]$$

Note: $\quad \underbrace{j \le}\ \nu < j+1$

Break.

Intuitively.

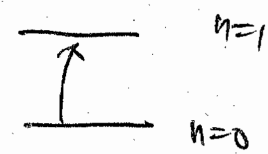


1) B very big, $\nu < 1$, $j = 0$

$$E = N\mu_B B_0 \frac{1}{\nu} = N\mu_B B$$

$$E = N\mu_B B_0 \qquad (\nu = 1)$$

2) $1 \leq \nu \leq 2$ $\qquad j = 1$



$$\frac{E}{N\mu_B B_0} = \frac{3}{\nu} - \frac{2}{\nu^2} \qquad \left(\text{e.g. } \nu = 2,\ \frac{E}{N\mu_B B_0} = 1\right)$$

Start oscillating.

at $\nu = 1$, ~~deco~~ decrease B further,

some electrons jump to next LL,

gaining energy, even though the energies

of those electrons in $n=0$ level decrease.

9.3.3 De Haas - Van Alfen effect.

$$M = -\frac{1}{A}\frac{\partial E}{\partial B} \qquad (j \leq \nu < j+1)$$

$$= -\frac{1}{A}\left[N\mu_B (2j+1) - \frac{N\mu_B}{\nu} 2j(j+1) \right]$$

$$= -(2j+1) n \mu_B + \frac{2n\mu_B}{\nu} j(j+1)$$

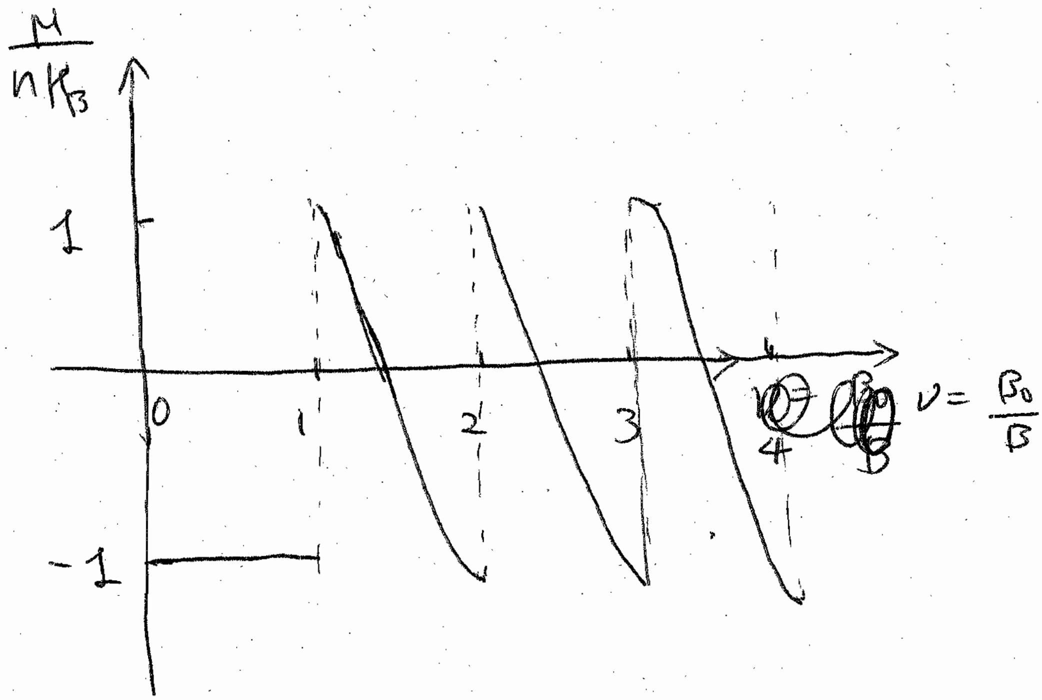
$$= -n\mu_B \left[(2j+1) - \frac{2}{\nu} j(j+1) \right] \qquad n = \frac{N}{A}$$

$$j \leq \nu < j+1$$

$$M(0<\nu<1) = -n\mu_B \qquad (j=0)$$

$$M(\nu=j) = n\mu_B$$

$$M(\nu=j+1) = -n\mu_B$$



Remarks:

1) M oscillates between $Q-n\mu_B$, and $n\mu_B$

2) M jumps at each integer value of filling ratio ν. $\Rightarrow$ trace fermi surface..

3) To understand the jump: (see energy plot)

at $\nu = n$

Oscillation of M with $\frac{1}{B}$: De Haas - Van Alfen effect!

4) We considered free electrons in 2d, in reality periodic potential and 3-dimensional

It turns out oscillations persist, and again give important information about the fermi surface (shape).

Oscillations also exist for other in $\left(\frac{1}{B}\right)$ quantities like specific heat and magnetic susceptibility.

Ashcroft and Mermin chapt. 14.

End of L. 6

9.4 Integer Quantum Hall effect:

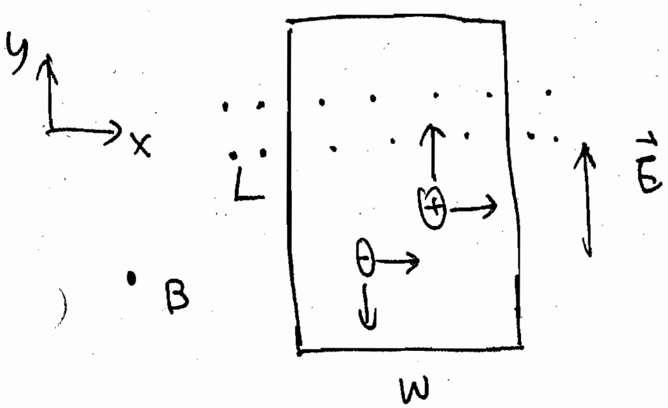
1980: Von Klitzing; 1985: Nobel prize

Plan:
1. Classical Hall effect
2. An idealized approximation to QHE
3. experiment

9.4.1 Classical Hall effect

- Set-up

Consider a strip of conductor lying in x-y plane



Apply a constant electric field E in y-direction:

- E induces an electric current in y-direction.

- The direction of the current is independent of the sign of the charge carrier.

Now apply a magnetic field B in $\hat{z}$-direction:

- under Lorentz force, ~~both~~ all charge carriers move toward $+x$ direction. $\Rightarrow$

- $+$ charged ~~particles~~ generate current in $+x$ direction

  $-$ . . . . . . a current in $-x$ direction

This current is called Hall current and depends on the sign of charge carriers. (an important application of Hall effect.)

$\frac{1}{2}$

Hall conductivity:

$$\sigma_H \equiv \frac{j_x}{E_y}$$

Longitudinal conductivity:

$$\sigma_L \equiv \frac{j_y}{E_y}$$

Let us now calculate σ_H, σ_L for free electron

~~To find σ_H and σ_L~~

We need ~~to understand~~ the motion ~~of the charged particles~~ a bit ~~more quantitatively~~.

Calculation of classical Hall conductivity..

- classical motion:

$$m \frac{d\vec{v}}{dt} = q \left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right)$$

$$\vec{E} = E \vec{e}_y \quad , \quad \vec{B} = B \vec{e}_z$$

We will consider $\underline{\frac{E}{B} \ll 1}$

(Note: $E = 1 \ \text{Volt/cm} = 2.3 \times 10^{-4} \frac{eV^2}{(\hbar c)^{\frac{3}{2}}}$

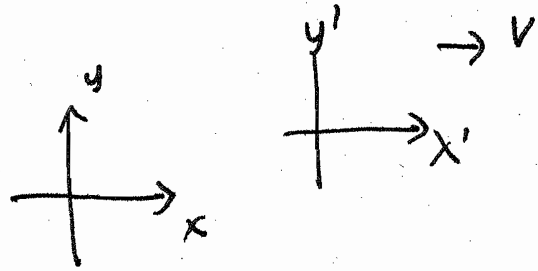
$B = 1 \ \text{gauss} = 6.9 \times 10^{-2} \frac{eV^2}{(\hbar c)^{\frac{3}{2}}}$)

$\Rightarrow$ Circular motion with angular velocity ω_L around a center, which moves with a velocity

$$\boxed{v_H = \frac{E}{B} c}$$ in $\underline{\underline{+x}}$ direction.

(Put aside) 

A quick derivation:



Consider a frame moving with a velocity $\frac{V}{c} \ll 1$, (along x direction)

then in this frame:

$$E_y' \doteq E_y - \frac{V}{c} B_z$$

$$B_z' \doteq B_z - \frac{V}{c} E_y \quad ,$$ all other components remain zero

Take $\underline{V = \frac{E}{B} c = V_H}$

$$\Rightarrow \quad E_y' = 0, \qquad B_z' = B\left(1 - \frac{V^2}{c^2}\right) \doteq B$$

$\Rightarrow$ Circular motion in the moving frame.

Q.E.D.

Note:

$$\boxed{Y} = Y' = y - \frac{v_x'}{\omega_L}$$

$$= y - \frac{1}{\omega_L}(v_x - V_H)$$ is a constant of motion.

• Hall conducitivity

Thus we have:

$$j_x = nqv_H = \frac{nqc}{B}E$$

n: density of of charged carriers

$$\Rightarrow \boxed{\sigma_H = \frac{nqc}{B}}$$

$j_y = 0$ since no net motion in y-direction.

$$\boxed{\sigma_L = 0}$$

In reality: charged particles often have collisions with ions, ~~are each other~~ defects or each other

$\sigma_L \neq 0$ (see Jaffe's notes)

But in a strong B field σ_L is small.

Break here.

9.4.2 A quantum charged particle in crossed $\vec{E}$ and B fields.

(start from script)

We now would to see what happens if we include Quantum effects.

We will first start with an idealized situation, ignoring all complexities of a physical conductor — electron interactions, electron–ion interaction, impurities etc.

$\Rightarrow$ Free charged particles in crossed E and B fields.

A: spectrum and wave function:

$$H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + q\phi$$

$$\vec{E} = E\,\vec{e}_y = -\nabla\phi$$

$$\vec{B} = B\,\vec{e}_z = \nabla\times\vec{A}$$

Gauge:

$$\phi = -Ey$$

$$\vec{A} = (-By, 0, 0)$$

(easiest)

other gauges much more complicated.

$$\Rightarrow \quad H = \frac{1}{2m} P_y^2 + \frac{1}{2m}\left(P_x + \frac{qB}{c} y\right)^2 \mp qEy$$

$$[P_x, H] = 0$$

Take $\psi = e^{ik_x x} f(y)$, and let $y_0 = \dfrac{-c\hbar k_x}{qB} = -\ell_0^2 k_x$

$$H = \frac{1}{2m} P_y^2 + \frac{1}{2} m\omega_L^2 (y - y_0)^2 \mp qEy$$

$$= \frac{1}{2m} P_y^2 + \frac{1}{2} m\omega_L^2 \left(y - y_0 \mp \frac{qE}{m\omega_L^2}\right)^2$$

$$\mp qEy_0 - \frac{1}{2}\frac{q^2E^2}{m\omega_L^2}$$

$$= \frac{1}{2m} P_y^2 + \frac{1}{2} m\omega_L^2 (y - y_0 - y_1)^2$$

$$\mp qE(y_0 + y_1) + \frac{1}{2}\frac{q^2E^2}{m\omega_L^2}$$

$$\boxed{\begin{aligned} y_0 &= -\ell_0^2 k_x \\ y_1 &= +\frac{qE}{m\omega_L^2} \end{aligned}}$$

H.O. with frequency ω_L centered at $y_0 + y_1$ plus additional ~~constant~~ energy shift.

$\frac{1}{2}$

~~full~~ break here.

Thus:

$$\varepsilon_n(k_x) = (n+\tfrac{1}{2})\hbar\omega_L - qE(y_0+y_1) + \frac{1}{2}\frac{q^2E^2}{m\omega_L^2}$$

$$\psi_n(x,y) = e^{ik_x x}\,\phi_n(y-y_0-y_1)$$

$$\underline{y_0 = -\frac{c\hbar k_x}{qB}} = -\ell_0^2 k_x\,, \qquad y_1 = \frac{qE}{m\omega_L^2} = \frac{1}{\omega_L}\frac{qE}{qB}c$$

$$= \frac{1}{\omega_L}V_H \quad \underline{\underline{(\text{put aside})}}$$

Remarks.

1) Center of motion in y-direction is now at

$$Y = y_0 + y_1 = -\frac{cP_x}{qB} + \frac{1}{\omega_L}V_H$$

$$\downarrow$$

$$= \underline{y - \frac{1}{\omega_L}V_x} + \frac{1}{\omega_L}V_H = \underline{\underline{y - \frac{1}{\omega_L}(V_x - V_H)}}$$

(as before in Landau gauge)

↑

exactly as the classical result.

2) Energy:

$$\mathcal{E}_n(k_x) = (n+\tfrac{1}{2})\hbar\omega_L - qEY + \frac{m}{2}\frac{q^2E^2}{\left(\frac{q^2B^2}{c^2}\right)}$$

$$= (n+\tfrac{1}{2})\hbar\omega_L - qEY + \tfrac{1}{2}mV_H^2$$

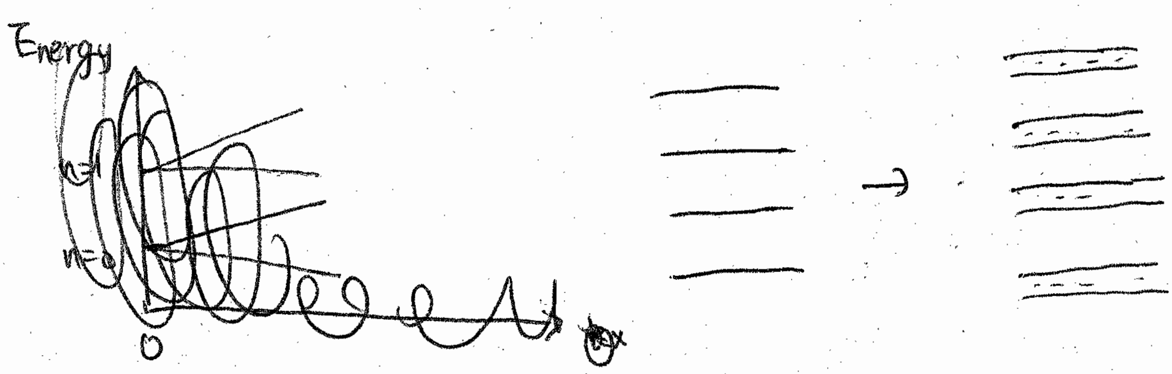
potential energy due to electric field at the center of motion

kinetic energy due to Hall velocity.

3) Degeneracy in k_x broken, since $y_0 \propto k_x$

$\Rightarrow$
$$\Delta\mathcal{E}_n = \mathcal{E}_n(k_x + \Delta k_x) - \mathcal{E}_n(k_x)$$

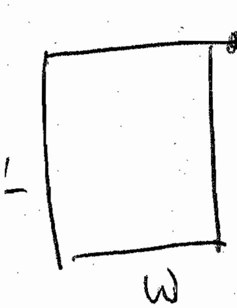
$$= +qE\ell_0^2\,\Delta k_x$$



14. $$\frac{d\varepsilon_n(k_x)}{dp_x} = v_H \quad \text{group velocity}$$

in x-direction.

15. ~~the~~ For a finite sample material with area A, the total number of states in a given Landau Level n is the same as before. (imposing periodic b.c. in x-direction)



$$\Delta Y = \Delta y_0 = \ell_0^2 \Delta k_x = \frac{2\pi \ell_0^2}{W} \quad \left(\begin{array}{l} y_1 \text{ is independent} \\ \text{of } k_x \end{array} \right)$$

$$D = \frac{L}{\Delta Y} = \frac{LW}{2\pi \ell_0^2} = \frac{\Phi}{\Phi_0} \qquad \left(\Phi_0 = \frac{hc}{e} \right)$$

B. Hall current:

In pset 3, we found ~~that~~ probability current

$$S_i = \frac{\hbar}{m} \mathrm{Im}(\psi^* \partial_i \psi) - \frac{q}{mc} \psi^* \psi A_i$$

$$= \mathrm{Re}(\psi^* v_i \psi)$$

(Local) Electric current density: $\vec{j} = q\vec{S}$

Macroscopically, we are interested in average current density.

e.g. $$J_x = \frac{1}{L} \int_0^L dy \; j_x = \frac{1}{L} \int_0^L dy \; q S_x$$

To compute S_i we need normalized wave functions.

$$\psi_{n k_x} = \frac{1}{\sqrt{W}} e^{i k_x x} \underbrace{\phi_n(y - y_0 - y_1)}_{}$$

real

$$k_x = \frac{2\pi m}{W} \qquad m \in \mathbb{Z}$$

$$\int_0^L dy \; \phi_n^2$$

ignore boundary effect. (ok when $L \gg \ell_0$)

$$\simeq \int_{-\infty}^{\infty} dy \; \phi_n^2(y) = 1$$

$$S_y = -\frac{i\hbar}{2m} \left(\psi^* \partial_y \psi - \psi \, \partial_y \psi^* \right) \qquad (A_y = 0)$$

$$= -\frac{i\hbar}{2m} \frac{1}{W} \left(\phi_n \, \partial_y \phi_n - \phi_n \, \partial_y \phi \right) = 0$$

$$S_x = \frac{-i\hbar}{2m} \left(\psi^* \partial_x \psi - \psi \, \partial_x \psi^* \right) - \frac{q}{mc} A_x \psi^* \psi$$

$$= \frac{1}{W} \phi_n^2 \cdot \left(\frac{\hbar k_x}{m} + \omega_L y \right)$$

$$= \frac{\omega_L}{W} (y - y_0) \; \phi_n^2 (y - y_0 - y_1)$$

Now average current density:

$$J_y = 0$$

$$J_x = \frac{1}{L}\int_0^L dy \quad q\, S_{ox}$$

$$= \frac{q}{L}\int_0^L dy \quad \frac{\omega_L}{W}(y-y_0)\,\phi_n^2(y-y_0-y_1)$$

$$= \frac{q\omega_L}{LW}\int dy \quad (y-y_0)\,\phi_n^2(y-y_0-y_1)$$

$$= \frac{q\omega_L}{LW}\Big[\int dy \quad (y-y_0-y_1)\,\phi_n^2(y-y_0-y_1)$$

$$+ \quad y_1\int \phi_n^2(y-y_0-y_1)\,dy \Big]$$

$$= \frac{q\omega_L}{LW}\, y_1$$

Note: $\int_0^L (y-y_0-y_1)\,\phi_n^2(y-y_0-y_1)$

$$\simeq \int_{-\infty}^{+\infty} (y-y_0-y_1)\,\phi_n^2(y-y_0-y_1)\,dy$$

$$= \int_{-\infty}^{+\infty} dx \; x\,\phi_n^2(x)$$

ignore boundary effect, $\phi_n^2(x)$ even in x

$$\int_0^L \phi_n^2(y-y_0-y_1) = 1$$

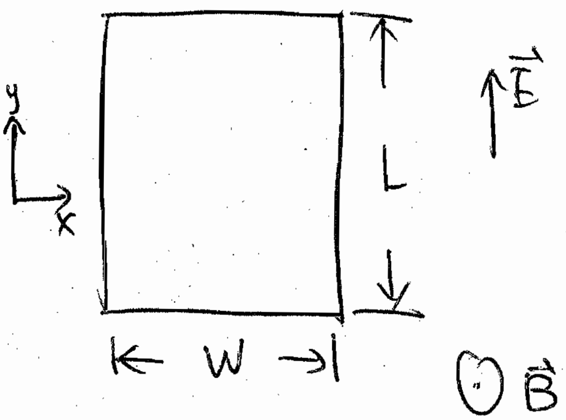
Thus we find:

$$J_x = \frac{e\omega_L}{LW} \cdot \frac{V_H}{\omega_L} = \frac{e V_H}{LW}$$

End of Lecture 7

lec. 8:

Last Lecture:



$$S_y = 0$$

$$S_x \neq 0$$

$$J_x = \frac{1}{L}\int_0^L dx \; q S_x$$

$$= \frac{q V_H}{LW}$$

$$V_H = \frac{E}{B} c$$

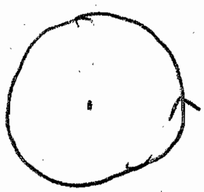
Remarks.

1) $J_x = 0$, $E = 0$ (since $U_H = \frac{E}{B} c = 0$)

despite that:

a) ψ is a plane wave in x-direction

b) S_x is nonzero



$E = 0$

$E \neq 0$

one sees that $\underline{\underline{p_x}}$ is not observable.

2) J_x is current density for a single particle.

Note: J_x $\underline{\underline{\text{independent}}}$ of $\underline{\underline{n}}$ and $\underline{\underline{k_x}}$.

J_x same for an electron in any LL.

3) N electrons:

independent of how electrons are filled.

$$J_H = N J_x = \frac{Ne}{A}\frac{E}{B}c$$

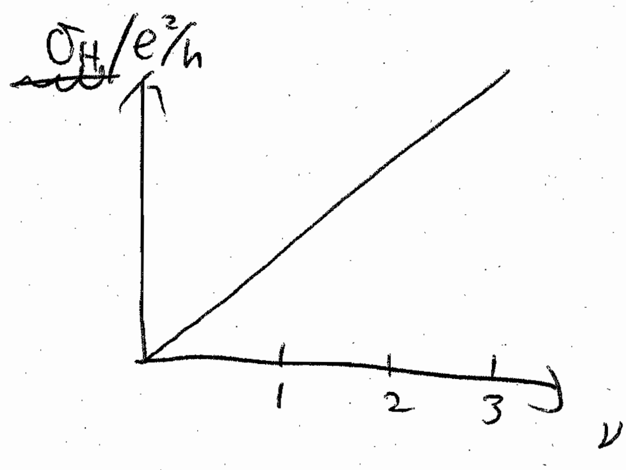
$$= \frac{nec}{B}E \qquad \left(n = \frac{N}{A}\right)$$

$$\Rightarrow \sigma_H = \frac{nec}{B} \qquad \text{(exactly classical answer)}$$

Recall.

$$\nu = \frac{N}{D} = \frac{N\Phi_0}{BA} = \frac{n\frac{hc}{e}}{B}$$

$$\Rightarrow \boxed{\sigma_H = \nu\frac{e^2}{h}}$$



4.5 Aharonov– Bohm effect:

Classically : $\vec{A}$, ϕ are convenient mathematical aid to compute the fields $\vec{E}$, $\vec{B}$.

QM : $\vec{A}$, ϕ are required to write down the Hamiltonian.

~~Nevertheless~~ but $\vec{A}$, ϕ are not gauge invariant

gauge invariant quantities are $\vec{E}$ and $\vec{B}$.

Does $\vec{A}$ and ϕ have independent significance ?

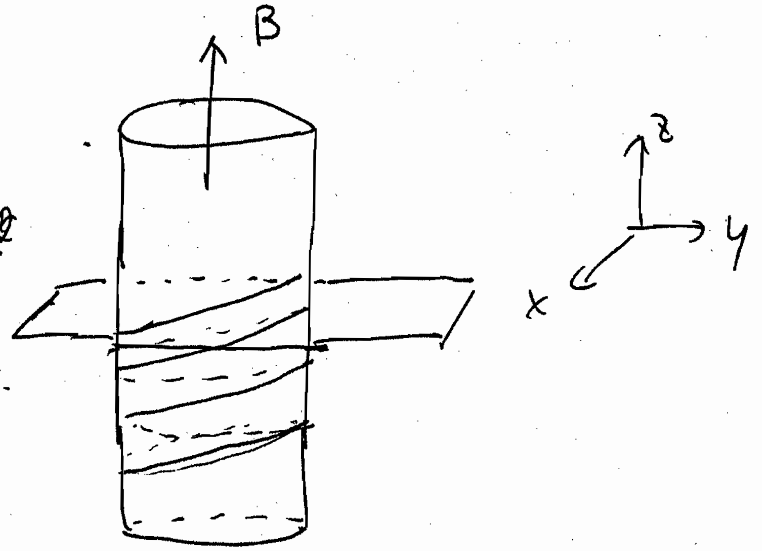
The answer is yes ! and beautifully illustrated by the A–B effect.

Consider a solenoid

with constant B inside

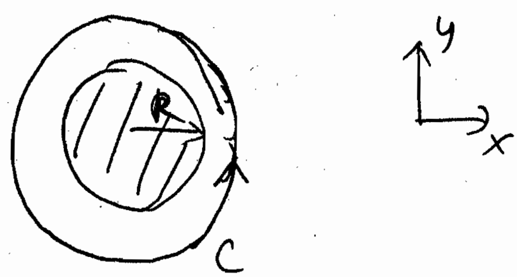
but $B=0$ outside.

(infinitely long)



In x-y plane :

$$\vec{B} = \begin{cases} B\,\vec{e}_z & r<R \\ 0 & r>R \end{cases}$$



For $r>R$, even though $B=0$, $\vec{A}$ cannot vanish.

consider integral

Stokes theorem

$$W_c = \oint_C \vec{A}\cdot d\vec{\ell} = \iint_{\text{region inside } C} d^2s \; \vec{e}_z \cdot \underbrace{(\nabla\times\vec{A})}_{=\vec{B}}$$

= flux through region bounded by C $= \underline{\underline{\pi R^2 B}}$

$\Rightarrow$ $\vec{A}$ cannot vanish outside the solenoid.

Thus:

Outside: $\vec{B} = \nabla \times \vec{A} = 0$

$$\vec{A} \neq 0$$

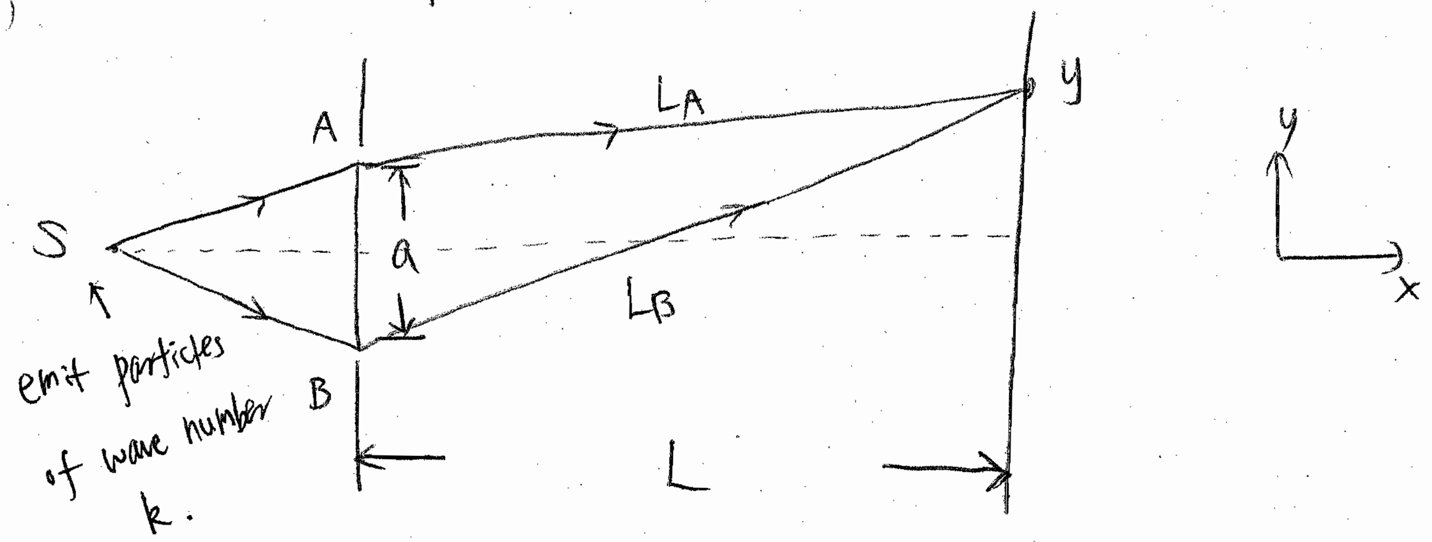
Can this non-vanishing $\vec{A}$ be observed?

or in other words have any physically observable effect?

answer: yes!

A-B effect (1954)

Double-slit experiment (review):



$$E = \frac{\hbar^2 k^2}{2m}, \qquad L_B - L_A \simeq \frac{a}{L} y \qquad (L \gg a, y)$$

k: wave number.

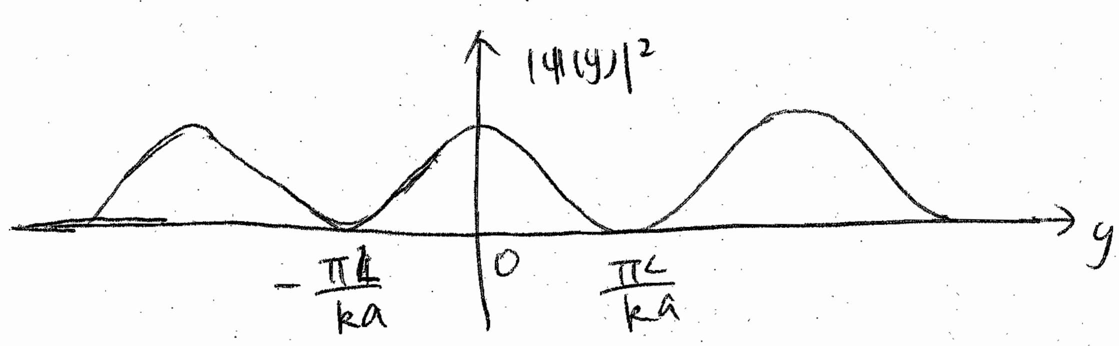
$$\psi(y) = \psi_A(y; S) + \psi_B(y; S)$$

Note: $|\psi_A| = |\psi_B| = \Lambda$ (different only by phases)

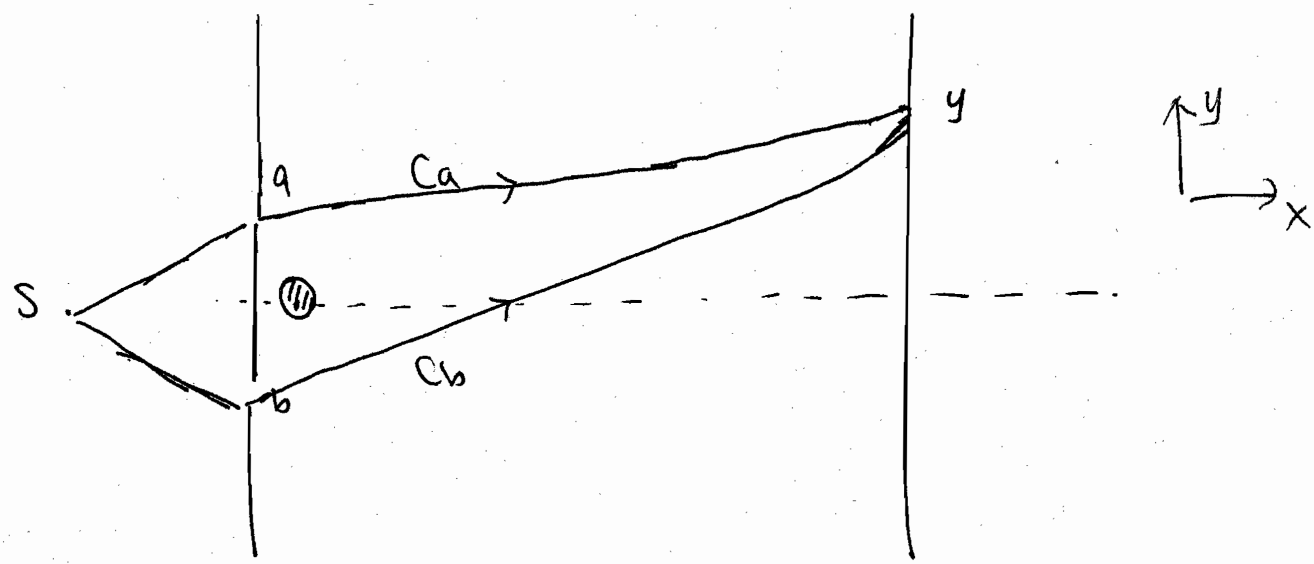
$$|\psi(y)|^2 = 2\Lambda^2 \left(1 + \cos k(L_B - L_A)\right)$$

interference

$$= 2\Lambda^2 \left(1 + \cos \frac{ka}{L} y\right)$$



10.6.2 Aharonov – Bohm experiment:



Now put a solenoid behind the screen. If the Solenoid is ideal, no magnetic field outside. ~~However, as we discussed earlier~~ while, $\vec{A} \neq 0$ ~~outside~~.

Does the nonzero $\vec{A}$ affect the ~~Now what are the~~ interference pattern?

For this purpose, ~~we need to solve how~~ Let us see how an electron propagates in ~~an arbitrary~~ this $\vec{A}$ field.

consider x–y plane.

The Schrodinger equation ($\vec{A}$: time-independent)

$$\frac{1}{2m}\left(-i\hbar\nabla - \frac{q}{c}\vec{A}\right)^2 \psi = E\psi \qquad (*)$$

Trick:

$$\psi(\vec{x}) = \underbrace{\exp\left(\frac{iq}{\hbar c}\int_{\vec{x}_0}^{\vec{x}} \vec{A}\cdot d\vec{\ell}\right)}_{g(\vec{x})} \psi_0(\vec{x}) \qquad (**)$$

$\vec{x}_0$ an arbitrary reference point.

Then

$$\left(-i\hbar\nabla - \frac{q}{c}\vec{A}\right)\psi = g(\vec{x})(-i\hbar\nabla)\psi_0$$

$$\left(-i\hbar\nabla - \frac{q}{c}\vec{A}\right)^2\psi = g(\vec{x})(-i\hbar\nabla)^2\psi_0$$

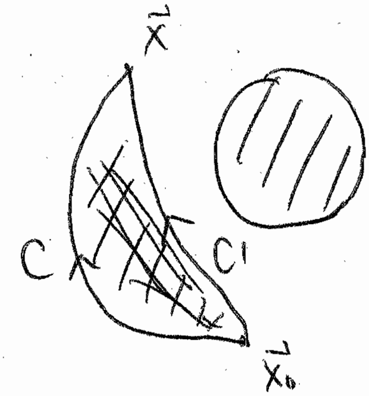
$(*) \Rightarrow$

$$\frac{1}{2m}\, g(-i\hbar\nabla)^2\psi_0 = E\cancel{g}\,\psi_0$$

$$\Rightarrow \quad -\frac{\hbar^2}{2m}\nabla^2\psi_0 = E\psi_0$$

ψ_0 : ~~satisfies~~ ~~the~~ free - particle ~~equation~~ wave function.

Note: for c and c' lying on the same side of the solenoid



$$\psi_c(\vec{x}) = \psi_{c'}(\vec{x})$$

(exercise for students)

Since :

$$\psi_c(\vec{x})\,\psi_{c'}^{-1}(\vec{x}) = \exp\left[\frac{iq}{\hbar c}\left(\int_c \vec{A}\cdot d\vec{x} - \int_{c'} \vec{A}\cdot d\vec{x}\right)\right]$$

$$= \exp\left[\frac{iq}{\hbar c}\oint_{c-c'} \vec{A}\cdot d\vec{x}\right]$$

$$= 1$$

$$\oint_{c-c'} \vec{A}\cdot d\vec{x} = \text{magnetic flux through shaded region}$$

$$= 0$$

Back to the double slit experiment:

$$\psi(y) = \psi_a(y;s) + \psi_b(y;s)$$

$$\psi_a(y;s) = \underbrace{\exp\left(\frac{iq}{\hbar c}\int_{C_a}\vec{A}\cdot d\vec{\ell}\right)}_{g_a}\psi_a^{(0)}(y;s)$$

$$\psi_b(y;s) = \underbrace{\exp\left(\frac{iq}{\hbar c}\int_{C_b}\vec{A}\cdot d\vec{\ell}\right)}_{g_b}\psi_b^{(0)}(y;s)$$

C_a: $s \to a \to y$ arbitrary trajectory

C_b: $s \to b \to y$

$$|\psi(y)|^2 = |\psi_a(y;s)|^2 + |\psi_b(y;s)|^2 + \left(\psi_a^*(y;s)\,\psi_b(y;s) + \psi_b^*(y;s)\,\psi_a(y;s)\right)$$

$$= |\psi_a^{(0)}|^2 + |\psi_b^{(0)}|^2 + \left(g_a^* g_b\, \psi_a^{(0)*}\psi_b^{(0)} + g_b^* g_a\, \psi_b^{(0)*}\psi_a^{(0)}\right)$$

Note:

$$g_a^* g_b = \exp\left(\frac{iq}{\hbar c} \int_{C_b - C_a} \vec{A} \cdot d\vec{l} \right)$$

$$= \exp\left(\frac{iq}{\hbar c} \oint_C \vec{A} \cdot d\vec{l} \right)$$

$$= \exp\left(\frac{iq}{\hbar c} W_c \right)$$

$$= \exp\left(\frac{iq \Phi}{\hbar c} \right)$$

$$\Rightarrow \quad |\psi(y)|^2 = 2|\lambda|^2 \left[1 + \cos\left(\frac{ka}{L} y + \frac{q\Phi}{\hbar c} \right) \right]$$

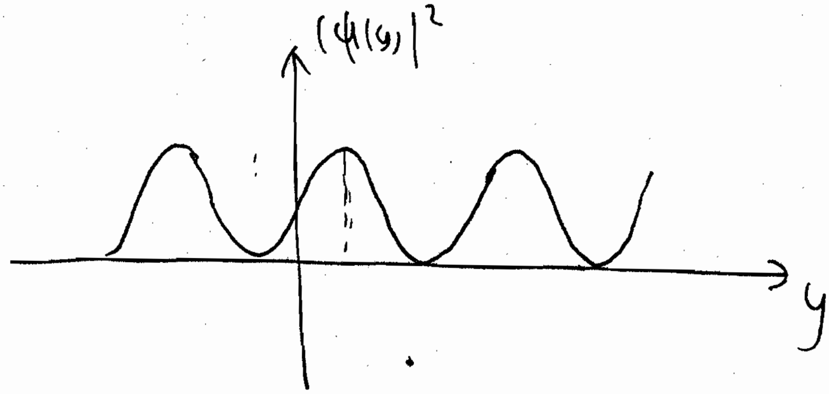
$$= 2|\lambda|^2 \left[1 + \cos\left(\frac{ka}{L} y + \frac{\Phi}{\Phi_0} 2\pi \right) \right]$$

$$\boxed{\Phi_0 = \frac{hc}{q}}$$

End of L8

Remarks:

1) Fringes shifted: depend on $\frac{\Phi}{\Phi_0}$



2) only when $\Phi = k \frac{hc}{q} = k\Phi_0$ $\quad k \in \mathbb{Z}$

The pattern does not change

3) Only fractional part of $\frac{\Phi}{\Phi_0}$ is observable.

Aharonov-Bohm effect.

Moral of the Story:

Even though $\vec{B} = \nabla \times \vec{A} = 0$ outside the solenoid,

$$W_C = \oint_C \vec{A} \cdot d\vec{\ell} \;\; = \;\; \text{flux through the region bounded by } C$$

$$\neq 0$$

for any C enclosing the solenoid.

W_C is observable through double-slit exp.

(gauge invariant) (exercise)

Note. C can be arbitrary far away.

topology!