8 Degenerate Fermi systems.

In 8.05:

wave function describing identical fermions must be anti-symmetric under the exchange of any two particles.

$\Rightarrow$ Pauli exclusion principle: no two fermions can occupy the same state.

an important application: atomic structure

(briefly discussed in 8.05)

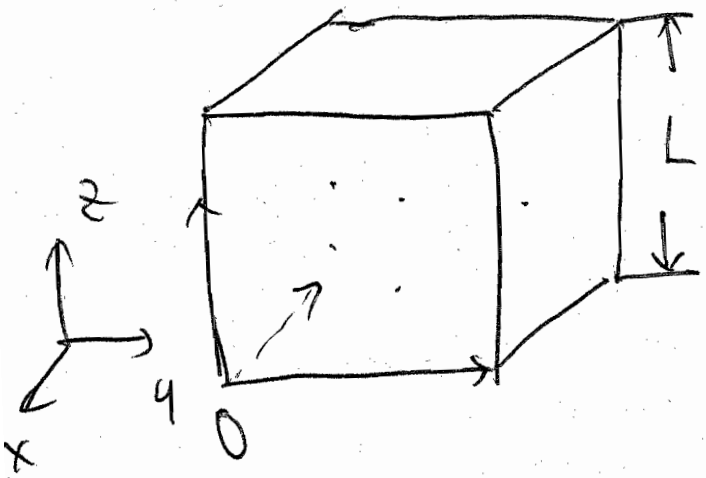
Here: we consider

= same for other applications in condensed matter and astrophysics.

We start by solving an idealized simple problem, and then consider its physical applications.

§1 Free electron gas:

Consider N non-interacting electrons confined inside a 3-d box, with length L on each side.



Would like to know:

1) Ground state wave function and energy

2) physical properties of the ground state.

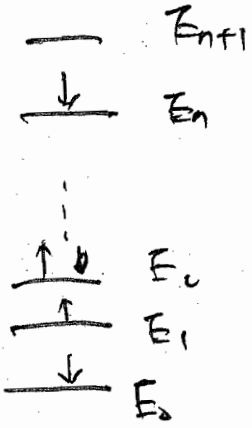
8.1.1

A Digression: (for review)

General strategy for solving problems involving multiple non-interacting fermions: (say electrons, spin $\frac{1}{2}$)

1) Solve single particle schrodinger equation.

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(x)\right)\psi_n = E_n \psi_n$$



2) Full multi-particle states are constructed by occupying electrons in different single-particle states (pauli principle).

including spin

3) Ground State: occupy lowest available states.

Note: Strategy does not work for interacting fermions,

in which case one must solve full problem

$$\left(-\frac{\hbar^2}{2m}\nabla_1^2 + \cdots - \frac{\hbar^2}{2m}\nabla_N^2 + V(x_1 \cdots x_N) \right) \psi(x_1 \cdots x_N)$$

$$= E\, \psi(x_1 \cdots x_N)$$

In reality: electrons always interact with each other.

Non-interacting electron is an approximation

when mutual interaction is not important.

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8.1.2

(60) Single-particle wave functions.

Single-particle wave functions inside the box: (3d infinite square well

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi$$

$$\psi\big|_{\text{boundary}} = 0$$

$$\Rightarrow \quad \psi(x,y,z) = \left(\frac{2}{L}\right)^{\frac{3}{2}} \sin k_1 x \sin k_2 y \sin k_3 z$$

$$k_i = \frac{n_i \pi}{L} \qquad i = 1, 2, 3$$

$$n_1, n_2, n_3 = 1, 2, \cdots \quad \text{(positive integers)}$$

$$E = \frac{\hbar^2}{2m}\left(k_1^2 + k_2^2 + k_3^2\right)$$

2½

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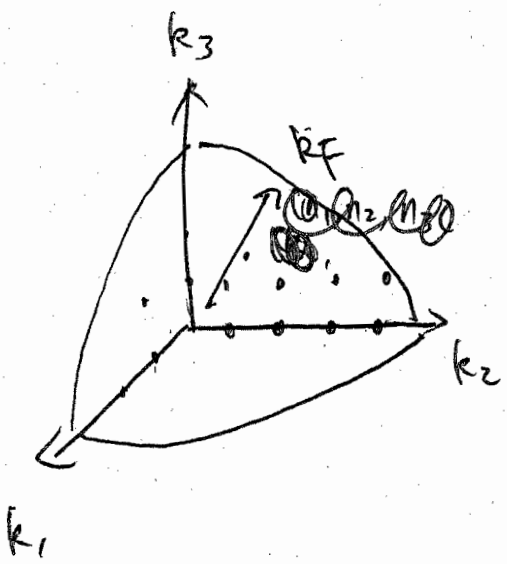
Visualization:

Single particle

State $\longleftrightarrow$ $(k_1, k_2; k_3)$: lattice points in the positive octant of $\vec{k}$-space.



Lattice Spacing: $\frac{\Pi}{L}$ in all directions

Note: 1) each state occupies a volume $\left(\frac{\Pi}{L}\right)^3$

2) Energy of a state is proportional to the distance of the point from the origin in $\vec{k}$-space

3) Each lattice points allow 2 electrons. $(\uparrow\downarrow)$

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Break

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8.1.3

Ground state energy:

Ground state: Choose $\frac{N}{2}$ points closest to the origin. (spin $\frac{1}{2}$)

When N is very large (say 10^{20})

- $\frac{N}{2}$ points can be considered as filling a $\frac{1}{8}$ ball in k-space, (draw)
- Points on the surface of the ball may or may not be filled. Their number is small compare to N, if the ball is big enough.

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- Denote the radius of the ball by k_F.

- Then

$$\frac{1}{8} \cdot \frac{4}{3}\pi k_F^3 = \frac{N}{2} \cdot \left(\frac{\pi}{L}\right)^3$$

$\frac{1}{8}$: positive octant

$\frac{N}{2}$: Spin-$\frac{1}{2}$

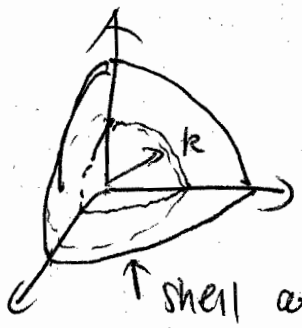
$\left(\frac{\pi}{L}\right)^3$: volume occupied by a single state

$$\Rightarrow \boxed{k_F = \left(\frac{3\pi^2 N}{L^3}\right)^{\frac{1}{3}} = \left(3\pi^2 \frac{N}{V}\right)^{\frac{1}{3}}}$$

the sphere with $|\vec{k}| = k_F$ is called Fermi surface.

$p_F = \hbar k_F$ is Fermi - momentum

$\epsilon_F = \frac{\hbar^2 k_F^2}{2m}$: Fermi energy



- Total ground state energy:

$$E_{tot} = \int_0^{k_F} \frac{4\pi k^2 dk \cdot \frac{1}{8}}{\left(\frac{\pi}{L}\right)^3} \cdot 2 \cdot \frac{\hbar^2 k^2}{2m}$$

$$= \frac{\hbar^2 V}{2\pi^2 m} \frac{k_F^5}{5}$$

$$= \frac{\hbar^2 V}{10\pi^2 m} \left(3\pi^2 \frac{N}{V}\right)^{\frac{5}{3}}$$

$\frac{q^2}{V}$

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Remarks:

(1) $\bar{\epsilon} = \frac{E_{tot}}{N} = \frac{\hbar^2 k_F^2}{2m} \frac{V k_F^3}{5\pi^2 N} = \frac{3}{5}\epsilon_F$

↑ average energy per particle

(2) $E_{tot} \propto N^{\frac{2}{3}} V^{-\frac{2}{3}}$

$V \downarrow$ $E \uparrow$, it costs energy to compress the electron gas

$\Rightarrow$ pressure.

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(3) Degenerate pressure.

$$P = -\left.\frac{dE_{tot}}{dV}\right|_N$$

$$= \frac{\hbar^2 (3\pi^2)^{\frac{2}{3}}}{5m} \left(\frac{N}{V}\right)^{\frac{5}{3}} \propto n^{\frac{5}{3}}$$

↑ density

5

(4) $P \propto \frac{1}{m}$

More massive particle needs larger # density (thus even larger mass density) to achieve the same pressure.

will see electron v.s. neutron.

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(5) When N is very large, the expressions for k_F, E_{tot}, P should not depend sensitively on the shape of the box.

(Bulk properties of a system does not depend sensitively on boundary conditions, shape)

End of lecture 1

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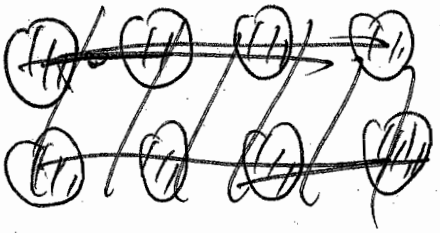
§1.4 An application : metallic properties of solids

(alkali metals)

Solids :

- atoms or molecules closed packed together and (typically) arranged in a periodic structure
- conductors, semi-conductors, insulators.



For each atom in the lattice :

loosely bound

can be detached

ion

E.g. Copper : one such electron per atom

# of atoms / cm^3 ~ 8.5×10^{22}

$\Rightarrow \frac{N}{V} \sim 8.5 \times 10^{22} / cm^3$ (per 1 #4)

$\frac{1}{4}$

lowest order approximation:

1) ignore the potential due to ions

2) ignore Coulomb interaction between electron

$\Rightarrow$ Free electron gas in a box (size of metal)

$\Rightarrow$ Sommerfeld electron gas theory

which explains ~~electric, thermal conductivities, spec~~ can be used to

~~metallic properties reasonably well~~ of various metals, such as electric and thermal conductivity, specific heat ...

Very important in early days of solid state physics.

we will not go into details here.

In pset 1, you will also check that:

1) treating electrons as non-relativistic is justified for metals (using example of Copper)

2) at room temperature, the material is "cold", i.e. the state of the electron gas is very close to g.s.

Ask yourself: how to check the other two assumptions.

In metals, electrons degeneate pressure does not play a major role. We will now examine an application that the pressure plays an essential role.

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8.2 White dwarf stars

8.2-1 Introduction

physics ~~underlying~~ are

life of an Ordinary star: (like sun)

- fusion ⟶ heat ⟶ thermal pressure

  ($4H \rightarrow {}^4He + 2e^+ + 2\nu_e$)

- Pressure balances gravity ⟶ star stable

- when H is gone, burns ${}^4He \rightarrow$ C, O

  Very heavy stars burn C, O, if not

- e.g. for sun, C, O cannot burn

  What will happen?

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⇒ No more heat, no more pressure

⇒ Collapse

⇒ Atoms crashed together, electrons squeezed out

Collapse stopped by degenerate pressure of the electron gas

⇒ White dwarf: force due to gravity is balanced by degenerate pressure of electrons confined inside the star.

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V.

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Q: Given the mass M_* of a white dwarf, what is its radius R_*, at which the star is stabilized?

A proper treatment is rather involved.

but somewhat crude

Here a simple, rather qualitative treatment.

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8.2.2 Radius of a white dwarf

Idea: 1) Assume the star has radius R,

find the total energy of the system:

$$E(R) = E_{grav}(R) + E_{electron}(R)$$

2) R_*: minimize $E(R)$ w.r.t. R

i.e. $\left.\frac{dE}{dR}\right|_{R_*} = 0$ (force balance)

$\left.\frac{d^2E}{dR^2}\right|_{R_*} > 0$ (stable balance)

Note: $\bar{F}_{grav} = -\frac{dE_{grav}}{dR}$, $\bar{F}_{elec} = -\frac{dE_{elec}}{dR}$. (Average force)

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- parameters:

N: total number of nucleons (proton + neutron

$N = \frac{M_*}{m_p}$ (m_p: proton mass, $m_p \simeq m_n$)

f: # of electrons per nucleon

$N_e = f N$: total # of electrons

$= f \frac{M_*}{m_p}$

- Assumptions: (will check later)

1) Star has uniform density ρ

$$\frac{4}{3}\pi \rho R^3 = M_*$$

2) electron gas non-interacting. } check it later

3) electrons non-relativistic

$1\frac{1}{2}$

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- Finding ~~E(R)~~ ~~Total~~ ① $E_{electron}(R)$

$$E_{electron}(R) = \frac{V\hbar^2}{10\pi^2 m_e}\left(3\pi^2 \frac{N_e}{V}\right)^{\frac{5}{3}}$$

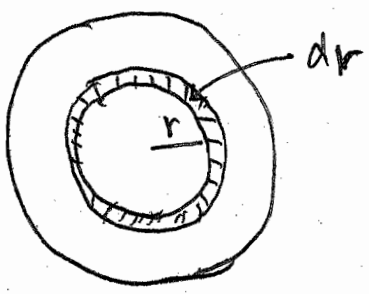
apply ↑ previous formula for a cubic box

$$= \frac{V\hbar^2}{10\pi^2 m_e}\left(3\pi^2 \frac{fN}{V}\right)^{\frac{5}{3}}$$

$$= \frac{k_1 N^{\frac{5}{3}}}{R^2} \qquad \left(k_1 = \frac{\hbar^2 f^{\frac{5}{3}}}{m_e}\frac{3}{10}\left(\frac{9\pi}{4}\right)^{\frac{2}{3}}\right) \quad (1)$$

② $P = -\frac{dE}{dR} > 0$ (want to increase R)

- Finding $E_{grav}(R)$



$$E_{grav}(R) = -\int_0^R \frac{Gm(r)}{r}\delta m(r)$$

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$m(r) = \frac{4}{3}\pi r^3 \rho$: mass enclosed by the shell

$\delta m(r) = 4\pi r^2 dr\, \rho$: mass of the shell

$$\rho = \frac{M_*}{\frac{4}{3}\pi R^3}$$

$$\Rightarrow \quad E_{grav} = -\int_0^R r^4 dr \cdot \frac{3GM_*^2}{R^6}$$

$$= -\frac{3GM_*^2}{5R} = -\frac{k_2 N^2}{R} \qquad (2)$$

$$k_2 = \frac{3Gm_p^2}{5}$$

$$F_{grav} = -\frac{\partial E_{grav}}{\partial R} < 0 \quad \text{(want to decrease } R\text{)}$$

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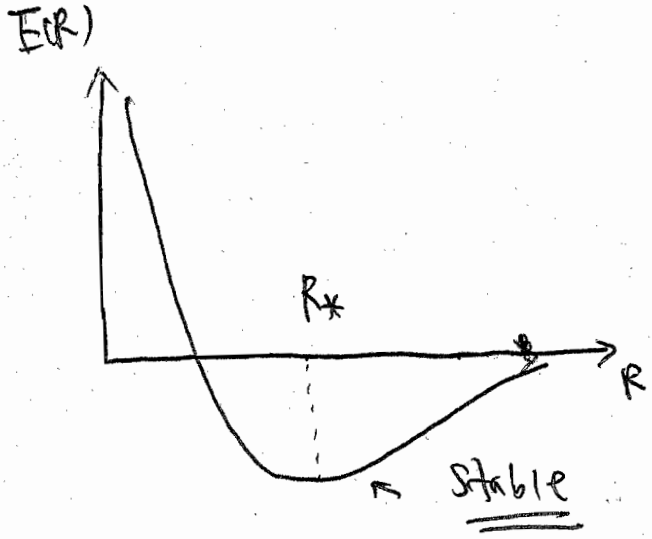
combining ① and ②

$$E(R) = -\frac{K_2 N^2}{R} + \frac{K_1 N^{\frac{5}{3}}}{R^2}$$



$$\frac{dE(R)}{dR} = 0 \Rightarrow$$

$$R_* = \frac{2K_1}{K_2} N^{-\frac{1}{3}}$$

$$\frac{2K_1}{K_2} = \left(\frac{9\pi}{4}\right)^{\frac{2}{3}} \frac{\hbar^2 f^{\frac{5}{3}}}{\mathcal{D} G m_e m_p^2}$$

2¼

Break.

High density

4) a teaspoon of white dwarf would weigh 5 tons.

a white dwarf with a Solar mass would be

about the size of the earth.

(from your pset 1)

5) ~~pset 1 check interaction~~

· Remarks:

1) $N \uparrow$, $M_* \uparrow \Rightarrow R_* \downarrow$

The more massive the star, the smaller radius.

Ordinary star: $M_* \uparrow \Rightarrow R_* \uparrow$

GOTO (2)

3) Notice: if $M_* \to \infty$, $R_* \to 0$

the density of star will become infinite?

In order to see whether we need to trust this conclusion, Check our assumptions: (non-interacting, non-relativistic)

both of which will likely to break down.

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4) Check non-relativistic assumption (~~check~~ of ~~non-interacting~~ see ~~pset 1~~)

Criterion: ok if $\frac{p_F}{m_e c} < 1$ or $\frac{\varepsilon_F}{m_e c^2} < 1$

$$p_F = \hbar k_F \qquad \varepsilon_F = \frac{\hbar^2 k_F^2}{2m_e}$$

$$\frac{\hbar k_F}{m_e c} = \frac{\hbar \left(\frac{3\pi^2 N_e}{V}\right)^{\frac{1}{3}}}{m_e c} = \frac{\hbar \left(\frac{3\pi^2 f N}{\frac{4}{3}\pi R_*^3}\right)^{\frac{1}{3}}}{m_e c}$$

(after plug in the expression for R_*) $=$

$$\left(\frac{4}{9\pi}\right)^{\frac{1}{3}} \frac{N^{\frac{2}{3}} G m_p^2}{\hbar c f^{\frac{4}{3}}} < 1$$

$\Rightarrow$

$$N < N_0 \equiv \left(\frac{9\pi}{4}\right)^{\frac{1}{2}} f^2 \frac{\left(\frac{\hbar c}{G}\right)^{\frac{3}{2}}}{m_p^3} \equiv N_0$$

$$= \left(\frac{9\pi}{4}\right)^{\frac{1}{2}} f^2 \left(\frac{M_{pe}}{m_p}\right)^3 \equiv N_0$$

↑ involve only M_{pe} and m_p

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$M_{pe} = \sqrt{\frac{\hbar c}{G}}$: <u>planck mass</u> <u>(see prof 1)</u>

$\Rightarrow$ $M_* = N m_p < \underline{M_\odot} = N_\odot m_p$

For a typical heavy element heavier than H

$f = \frac{1}{2}$ (e.g. He, C, O ...)

$M_{pe} = \cancel{1.2 \times 10^{19}}$ $\frac{GeV}{c^2}$

$M_p = \frac{1\,GeV}{c^2} = 1.78 \times 10^{-24}\ g$

$\Rightarrow$ $M_\odot \simeq 2 \times 10^{33}\ g$

$\simeq M_\odot$!

Thus for $M_* > M_\odot$ should use relativistic electrons.

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§3 Relativistic electron gas and the Chandrasekhar mas

Relativistic electrons:

$$E_{kin} = \sqrt{m_e^2 c^4 + \hbar^2 k^2 c^2}$$ (only different)

Ultra-relativistic: $\hbar c k \gg m_e c^2$

$$E_{kin} \approx \hbar c k + \dots$$

Let us see what happens if we assume most electrons are ultra-relativistic:

$$E_{electron} \approx 2 \int_0^{k_F} \frac{4\pi k^2 dk}{8 \left(\frac{\pi}{L}\right)^3} \cdot \hbar k c$$

3 1/4

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$\Rightarrow$

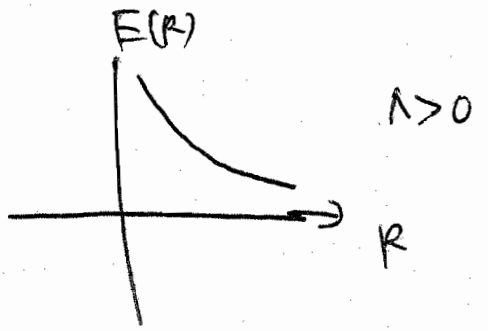
$$E_{elec} = \frac{V\hbar c}{4\pi^2} k_F^4 = \frac{V\hbar c}{4\pi^2} \left(3\pi^2 \frac{Ne}{V}\right)^{\frac{4}{3}}$$

power change

$$= \frac{k_3}{R} N^{\frac{4}{3}} \qquad \left(k_3 = \frac{3}{4}\left(\frac{9\pi}{4}\right)^{\frac{1}{3}} f^{\frac{4}{3}} \hbar c \right)$$

Now $$E_{tot} = -\frac{k_2 N^2}{R} + \frac{k_3}{R} N^{\frac{4}{3}}$$

$$= \frac{\Lambda}{R}$$

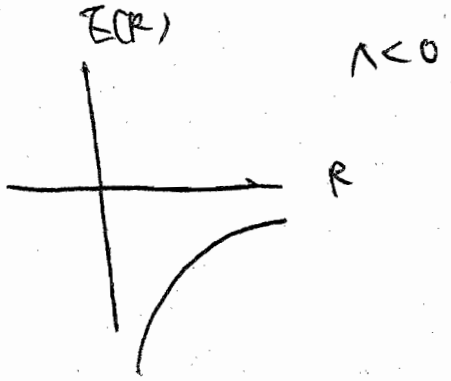


Ask yourself what happens for $\Lambda > 0$

If $\Lambda < 0$, $k_2 N^2 > k_3 N^{\frac{4}{3}}$

i.e. $k_2 N^2 > k_3 N^{\frac{4}{3}}$

or $N > \left(\frac{k_3}{k_2}\right)^{\frac{3}{2}} \equiv N_c$



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$$N_C = \left(\frac{K_3}{K_2}\right)^{\frac{3}{2}} = \left(\frac{5}{4}\right)^{\frac{3}{2}} \left(\left(\frac{9\pi}{4}\right)^{\frac{1}{3}} f^{\frac{4}{3}} \frac{\hbar c}{G m_p^2} \right)^{\frac{3}{2}}$$

$$= 3.7 f^2 \left(\frac{M_{pe}}{m_p} \right)^3 \cdot$$

$$\simeq 1.4\, N_0$$

$$\Rightarrow \quad M_C = 1.4\, M_\odot \qquad \text{Chandrasekhar mass.}$$

When $M_* > M_C \Rightarrow$ Collapse

Two possibilities:

1) $e + p \rightarrow n$

neutron star, supported by the degenerate neutron pressure.

2) $M > M_C^{(n)}$ (for neutron ●, ◎ about $3\, M_\odot$)

Collapse to form a black hole.

A little bit history.

1862. A. Clark discover Sirius B

$T = 25,000$ K

much fainter 10,000 than Sirius A

$\Rightarrow$ small radius

$\Rightarrow$ mass the same as Sun

$\Rightarrow$ extremely dense

Start here

$\rightarrow$ 1920 Eddington suggested high densities were due to complete ionization of the atoms in the interior.

1926 R.H. Fowler: electron degenerate gas, a few months after Fermi-Dirac statistics

1930: Chandrasekhar, @ age 19 on a ship to England, applied relativistic theory to Fowler's results ⇒ M_{chand}

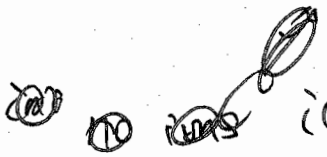
It took long time for people to be convinced of his results, in particular Eddington.

(Chandrasekhar himself did not know what to expect when the Mc is exceeded.

Chandrasekhar got 1983 Nobel prize.

8.4 Electrons in a periodic potential.

Last Lecture, metal: lattice ions surrounded by free electron gas.



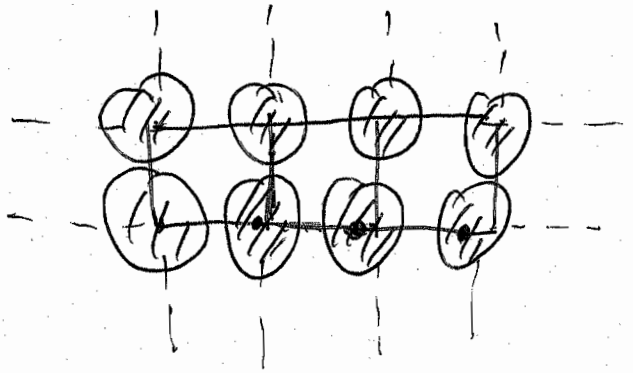
ignore: ion potential

mutual coulomb interact.

Now Let us examine whether ignoring ion potential is reasonable or not.

be inclined to conclude

If you think a bit about it, one would ~~have great difficulty~~ the opposite: electrons should have ~~would have~~ great difficulty going around.



ions: Closely packed.

radius of ion: a few angstrom

lattice spacings: a few angstrom

It turns out neither intuitions (a) ignore potential / (b) electrons have difficulty moving around is quite right.

We will see that, when including the ion potentials: (electrons still non-interacting)

due to magic of QM 1) electrons can still move around smoothly and easily, almost as free. (Bloch theorem) (b) is wrong.

2) Single-particle wave functions have qualitative difference from free electron gas. (Band structure)

⇒ conductor, semiconductor, insulator.

8.4.1 Bloch theorem.

Consider a single electron moving in a lattice of ions. Let us consider a 1-d version for simplicity.

potential is periodic; $U(x) = U(x+a)$

Single-particle schrodinger equation:

$$\left(-\frac{\hbar^2}{2m}\frac{d}{dx^2} + U(x)\right)\psi(x) = E\psi(x) \qquad (*)$$

Bloch theorem: $\psi(x) = e^{ik\cdot x}u_k(x)$

k real: $u_k(x) = u_k(x+a)$

Proof: (i) Define an operator $\hat{T}_a$ such that

$$\hat{T}_a \psi(x) = \psi(x+a) \qquad (1)$$

e.g. $T_a = e^{a\frac{d}{dx}}$ $\qquad T_a^+ = T_{-a} = (T_a)^{-1}$ (2)

unitary operator.

since $U(x+a) = U(x)$

$$[\hat{T}_a, H] = 0$$

$\Rightarrow$ we can choose eigenstates of H that are also eigenstates of $\hat{T}_a$, i.e.

$$\hat{T}_a \psi(x) = C(a)\, \psi(x) \qquad (3)$$

(ii) determine $C(a)$

Acting $\hat{T}_a$ n times.

On the one hand, (from ③)

$$\hat{T}_a^n \psi(x) = C^n(a)\, \psi(x)$$

On the other hand, (from ②)

$$\hat{T}_a^n = T_{na}$$

③ $\Rightarrow$ $\hat{T}_a^n \psi(x) = C(na)\, \psi(x)$

$\Rightarrow$ $C^n(a) = C(na)$

$\Rightarrow$ $\boxed{C(a) = e^{ika}}$ ④ for some constant k

Since $T_a^\dagger = T_a^{-1}$ $\Rightarrow$ k real.

$e^{-ik^*a} = e^{-ika}$

(iii) Equation ③ ①:

$$\hat{T}_a \psi = \psi(x+a) = e^{ika}\, \psi(x) \quad ⑤$$

Now Let

$$\psi(x) = e^{ikx}\, u_k(x)$$

eq (5) $\Rightarrow$

$$\Rightarrow \quad u_k(x+a) = u_k(x)$$

Q.E.D.

Remarks:

1. physical implication:

Recall free particle $\psi(x) = Ce^{ikx}$

$$|\psi(x)|^2 = \text{const}$$

Periodic potential: $\psi(x)$ is a plane wave modulated by a periodic function profile $u(x)$.

$$|\psi|^2 = |u(x)|^2 \quad \text{is periodic}$$

$\Rightarrow$ Wave functions are delocalized.

$\Rightarrow$ electrons can move around despite the presence of closely packed ions.

2. Range of values of k.

$$\hat{T}_a \psi_k(x) = e^{ika} \psi_k(x)$$

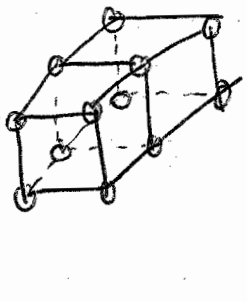
Since e^{ika} is periodic in ka,

we can restrict k to range

$$-\frac{\pi}{a} < k \leq \frac{\pi}{a}$$

3. 3-dimension generalization is immediate.

$$\psi(\vec{x}) = e^{i\vec{k}\cdot\vec{x}}\, u(\vec{x})$$

E.g. for a  cubic lattice

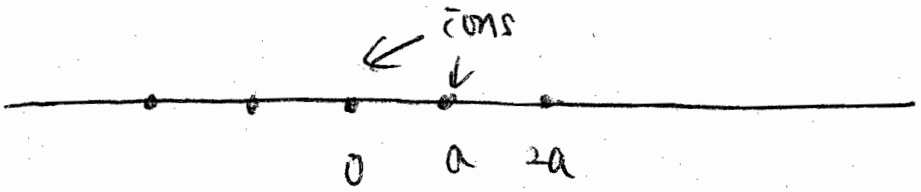
$$k_i \in \left(-\frac{\pi}{a}, \frac{\pi}{a}\right]$$

Different lattice structure $\Rightarrow$ different values to $\vec{k}$ (Brillouin zone)

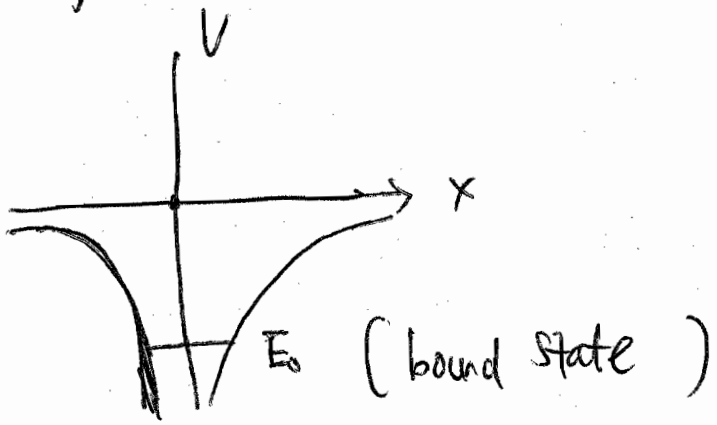
§4.2 Band structure (tight-binding model)

While Bloch theorem tells us that electrons can move around in a solid, it does not give us a direct physical intuition of how this happens. To get more physical intuition, let us now look at a simple model. From this model, we also learn another important effect of a periodic potential, (write section title)

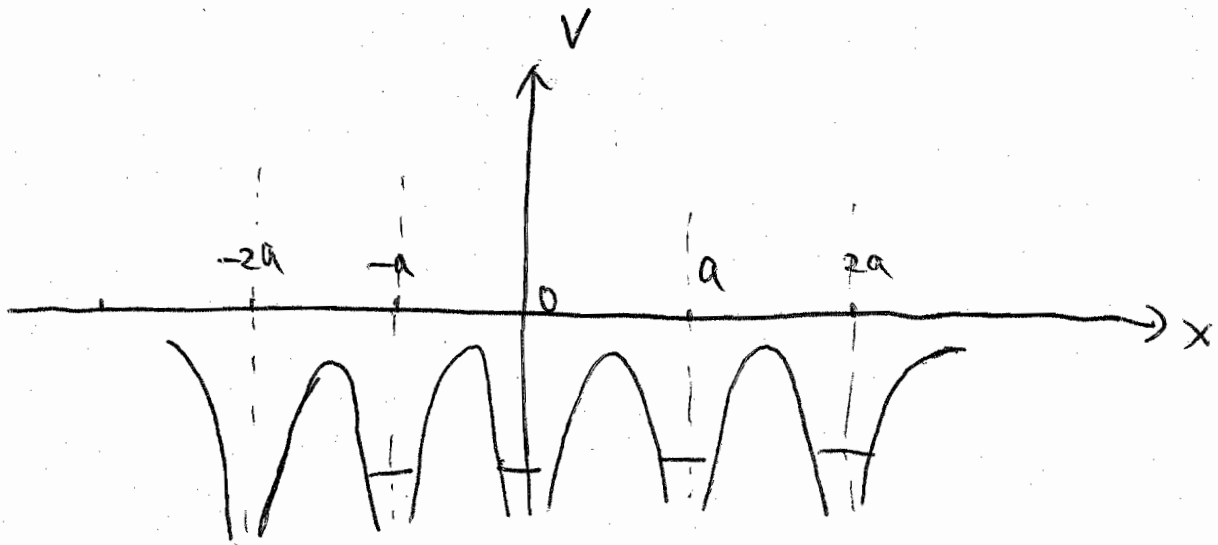
Consider a one-dimensional lattice (at the moment infinite)



potential of a single atom:



Full lattice potential:



What are the energy eigenstates?

Directly solving Schrodinger equation with such a potential is very difficult, and not necessary.

Want to know: How does the presence of other ions affect state E_0?

e.g. effect: an electron can tunnel from one atom to a neighboring one,

e.g. $x=0 \longrightarrow x = \pm a$

Let $|n\rangle$ to be the state that the electron is trapped by the atom at $x = na$.

assuming $\langle m|n\rangle = \delta_{nm}$, $\sum_n |n\rangle\langle n| = 1$

Ignore effects of other atoms:

$$H_0 = \sum_{n=-\infty}^{+\infty} E_0 |n\rangle\langle n|$$

in the basis of $|n\rangle$. (a complete set)

Now taking into account of tunneling:

$$H = \sum_{n=-\infty}^{+\infty} \left[E_0 |n\rangle\langle n| - \Delta |n+1\rangle\langle n| - \Delta |n-1\rangle\langle n| \right]$$

tunneling to neighboring atoms

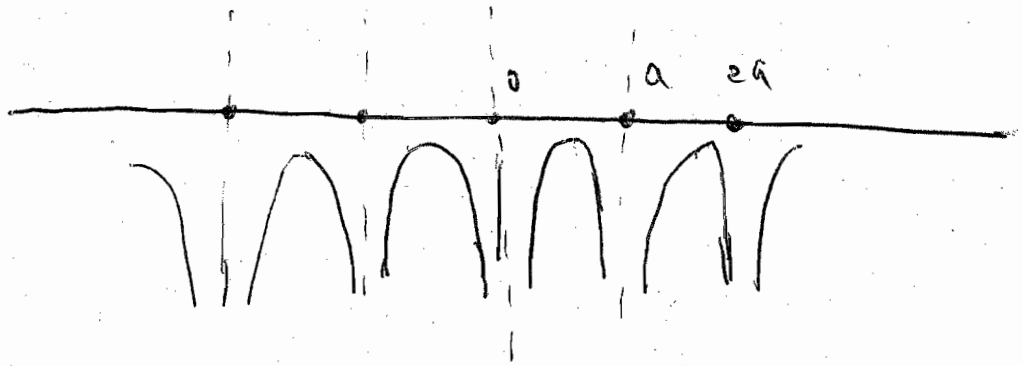
Now solve

$$H\psi = E\psi$$

End of lect. 3.

Lec. 4.

§. 4.2 Band structure (continued)



Mathematical idealization:

1) $|n\rangle$: states localized at $x = na$.

2) $\langle n|m\rangle = \delta_{nm}$

3) An electron at $x = na$ can hop to

$$x = (n \pm 1)\, a \qquad \text{(tunnelling)}$$

Convenient to write the Hamiltonian in $|n\rangle$ basis

The translation operator is

$$T|n\rangle = |n+1\rangle \quad \text{or} \quad T = \sum_{n=-\infty}^{+\infty} |n+1\rangle\langle n|$$

$$T^{-1} = \sum_{n=-\infty}^{+\infty} |n\rangle\langle n+1|$$

• Since H does not depend on the labelling

$$[H, T] = 0$$

In fact:

$$H = E_0 \mathbb{1} - \Delta(T + T^{-1})$$

Since $|n\rangle$ is a complete set.

Bloch theorem: Choose eigenstates of T

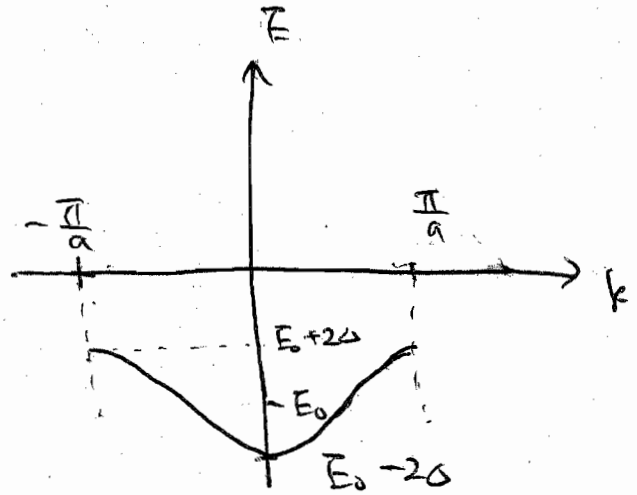
$$T|\psi_k\rangle = e^{ika}|\psi_k\rangle$$

$$\Rightarrow \quad H|\psi_k\rangle = E_k|\psi_k\rangle$$

With

$$E_k = E_0 - \Delta\left(e^{ika} + e^{-ika}\right)$$

$$= E_0 - 2\Delta\cos ka$$



$$-\frac{\pi}{a} < k \leq \frac{\pi}{a}$$

Remarks:

1) In pset 2 you will find $|\psi_k\rangle$.

and see explicitly $|\psi_k\rangle$ is delocalized.

Skip:
Intuitively: even though tunneling rate is typically very small, constructive interference due to periodic structure.

total # of states in a solid:

2. Zn reality, solid is not infinite.

We consider N sites with periodic boundary conditions on the wave function.

i.e, $L = Na$, $\psi(0) = \psi(L)$

$\Rightarrow$ $T^N |\psi_k\rangle = |\psi_k\rangle$

$T^N |\psi_k\rangle = e^{iNka} |\psi_k\rangle = |\psi_k\rangle$

energy eigenstates.

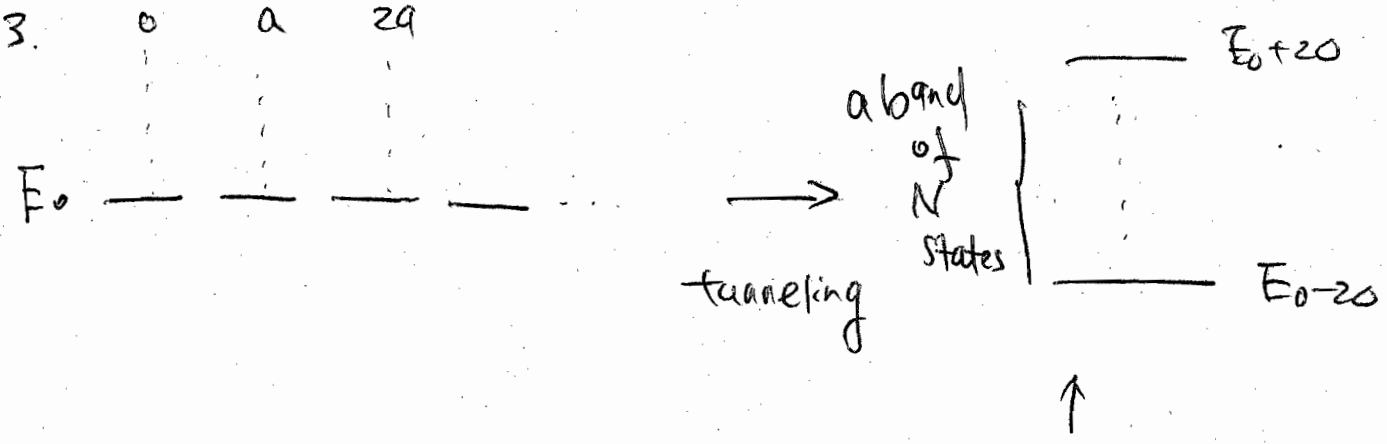
$\Rightarrow$ $Nka = 2\pi n$

$\Rightarrow$ $k = \frac{2\pi n}{Na}$, $-\frac{N}{2} < n \leq \frac{N}{2}$

total N states.

Effect of tunneling:

3. 0 a 2a

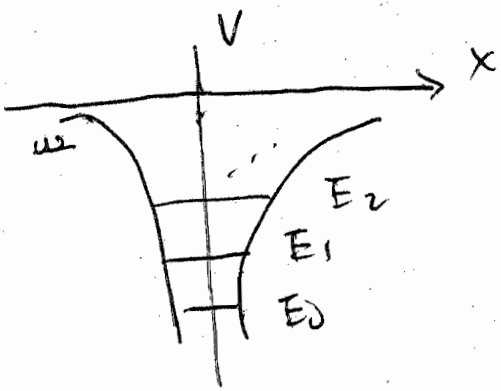


each state is delocalized

with equal probability at any site.

4. Band structure

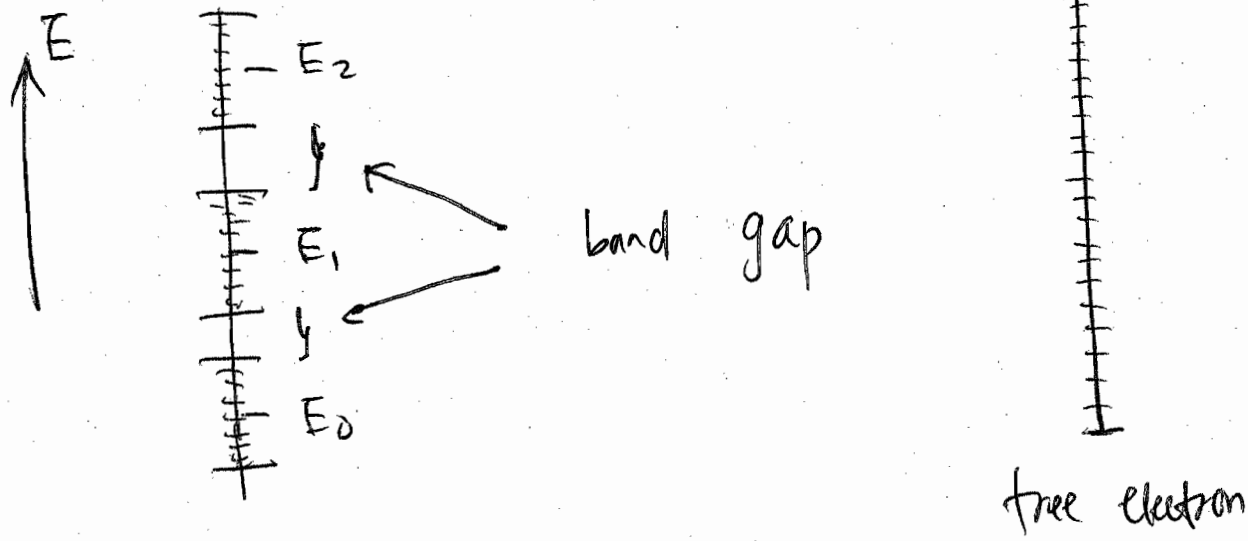
A single atom has more than one energy levels



Δ is usually much smaller than the energy differences

(Simplest situation)

Then in a lattice:



~~Band Zn~~

Zn more complicated situations, different bands can cross ...

5. conductors, semi-conductors and insulators.

Now conside M electrons. fill single-particle states.

Each band can hold 2N electrons.

conductor.

Ground state: no net current.

Conductors: partial filled

semi-conductor: ← Smaller band gap

insulator } filled