Quantum Physics III (8.06) Spring 2007
Solution Set 5

March 13, 2007

1. **The Aharonov-Bohm effect on energy eigenvalues (20 points)**

(a) **(2 points)** Using Stokes' theorem, $\oint \vec{A} \cdot d\vec{l} = \int \vec{B} \cdot d\vec{a}$, we find for a circular loop with radius $r < a$

$$2\pi r A_\phi = B_0 \pi r^2,$$

so within the cylinder we find $A_\phi = \frac{1}{2} B_0 r$. If $r > a$, Stokes' theorem tells us

$$2\pi r A_\phi = B_0 \pi a^2,$$

so $A_\phi = B_0 a^2/(2r)$ outside the cylinder.

(b) **(2 points)** By assumption, the particle can only move on a ring, so only p_ϕ appears in H and the radius is fixed to be b. Thus for our case the Hamiltonian is

$$H = -\frac{\hbar^2}{2m}\frac{1}{b^2}\left(\frac{\partial}{\partial\phi} - \frac{iq\Phi}{2\pi\hbar c}\right)^2,$$

where I have used $A_\phi = \Phi/(2\pi b)$.

(c) **(2 points)** The Schrodinger equation we would like to solve is

$$-\frac{\hbar^2}{2mb^2}\left(\frac{\partial}{\partial\phi} + \frac{i\Phi}{\Phi_0}\right)^2 \psi = E\psi,$$

where $\Phi_0 = \frac{hc}{|q|}$ and we take only the positive value of the charge (Note that for an electron $|q| = -q = e$). Putting $\psi = e^{in\phi}$, for n an integer to satisfy the boundary conditions on ψ in this gauge, we find $\frac{\hbar^2}{2mb^2}(n + \Phi/\Phi_0)^2\psi = E\psi$, and thus

$$E = \frac{\hbar^2}{2mb^2}\left(n + \frac{\Phi}{\Phi_0}\right)^2.$$

(d) **(3 points)** We must always have $\oint \vec{A} \cdot d\vec{l} = \Phi$ for a circular path surrounding the cylinder. Now, if $\vec{A} = \vec{\nabla} f$, $\int_a^b \vec{A} \cdot d\vec{l} = f(b) - f(a)$. We can consider breaking the circular path into two pieces,

$$\Phi = \int_{-\pi+\epsilon}^{\pi-\epsilon} r A_\phi \, d\phi + \int_{\pi-\epsilon}^{-\pi+\epsilon} r A_\phi \, d\phi.$$

When we take the limit $\epsilon \to 0$, the second term on the right hand side disappears because ϕ is oriented in the counter-clockwise direction. The first term can be written in terms of f, giving

$$\Phi = f(r, \pi - \epsilon) - f(r, -\pi + \epsilon)$$

in the limit $\epsilon \to 0$. The simplest choice for $f(\phi)$ is

$$f = \frac{\phi}{2\pi}\Phi. \tag{1}$$

(e) **(3 points)** We make the gauge transformation $\vec{A} \to \vec{A} - \vec{\nabla} f = \vec{A}'$, $\psi \to e^{-iqf/(\hbar c)}\psi = \psi'$, using the f of equation (1). The Schrodinger equation in the new gauge is

$$-\frac{\hbar^2}{2mb^2}\frac{\partial^2}{\partial\phi^2}\psi' = E\psi'. \tag{2}$$

This is a free particle Schrodinger equation, but it is only valid in the region $-\pi+\epsilon < \phi < \pi-\epsilon$ (as f is only defined in that region).

(f) **(3 points)** First, we express $\psi'(\pm(\pi-\epsilon))$ in terms of $\psi(\pm(\pi-\epsilon))$:

$$\begin{aligned}
\psi'(\pi-\epsilon) &= e^{-iq\Phi(\pi-\epsilon)/(2\pi\hbar c)}\psi(\pi-\epsilon) \to e^{-iq\Phi/(2\hbar c)}\psi(\pi-\epsilon) \\
\psi'(-(\pi-\epsilon)) &= e^{iq\Phi(\pi-\epsilon)/(2\pi\hbar c)}\psi(-(\pi-\epsilon)) \to e^{iq\Phi/(2\hbar c)}\psi(-\pi+\epsilon).
\end{aligned}$$

Thus

$$\psi'(\pi) = e^{-iq\Phi/\hbar c}\psi'(-\pi), \tag{3}$$

so the gauge transformed wave function is not single valued at $\phi = \pi$.

(g) **(3 points)** The Schrodinger equation (2) is a free particle equation, so we expect the solutions to be $\psi'(\phi) = e^{ik\phi}$. Using this guess for ψ', we find

$$\frac{\hbar^2 k^2}{2mb^2}\psi' = E\psi'.$$

The boundary condition (3) determines the allowed values of k:

$$e^{ik\pi} = e^{-iq\Phi/\hbar c}e^{-ik\pi},$$

so $2\pi(k+n) = \frac{-q\Phi}{\hbar c}$ for any integer n. Therefore, $k = \frac{-q\Phi}{2\pi\hbar c} + n = \frac{\Phi}{\Phi_0} + n$, and the allowed energy eigenvalues are

$$E = \frac{\hbar^2}{2mb^2}\left(\frac{\Phi}{\Phi_0} + n\right)^2. \tag{4}$$

These are precisely the same energies as those found in the original gauge, equation (1).

(h) **(2 points)** A plot of the energy eigenvalues as a function of Φ is shown in figure 1. In figures 2 and 3, the lowest and second lowest eigenvalues have been plotted as a function of Φ. As we increase Φ, some energies increase while others decrease; however, the energy levels intersect in such a way that the lowest, second-lowest, etc., energies are periodic in Φ, with period $\Phi_0 = \frac{2\pi\hbar c}{e} = \frac{2\pi e}{\alpha} = 2\pi \times 137e$, the flux quantum. There is no effect from the magnetic field when the flux enclosed by the particle is a multiple of the flux quantum, or in other words, if $B_0 = 0, \Phi_0/(\pi a^2), 2\Phi_0/(\pi a^2), \ldots$. The overall energy spectrum is insensitive to the integer part of Φ/Φ_0 due to the periodicity; we cannot tell the difference between the presence of one flux quantum and the presence of 2 million.

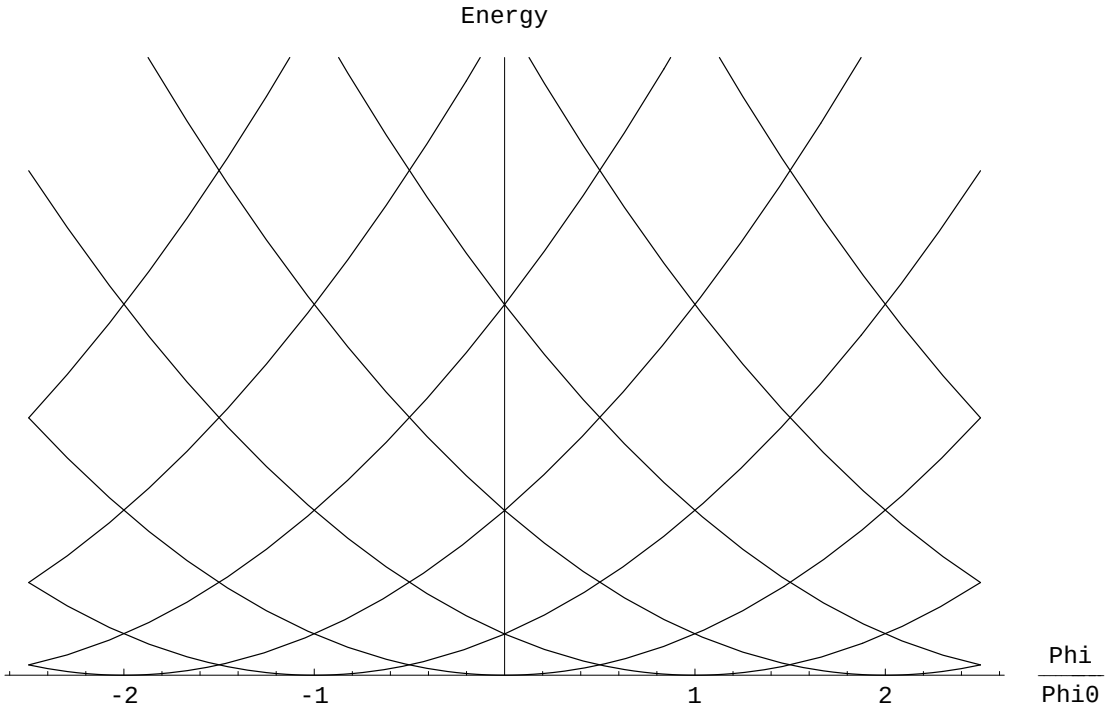

Figure 1: The energy spectrum as a function of Φ.

2. **The fractional quantum Hall effect (5 points)**

   As we increase the flux going through the solenoid, we increase the magnetic field, and thus the vector potential is time-dependent.

   This in turn induces an electric field, the magnitude of which is $E_\phi = -\frac{1}{c}\frac{\Delta A_\phi}{\Delta t} = -\frac{\Delta\Phi}{\Delta t}\frac{1}{2\pi rc}$. The azimuthal field E_ϕ in addition to azimuthal current creates radial current flow due to Hall effect. The radial current is $j = \frac{\Delta Q}{\Delta t \Delta l} = \sigma E$. Integrating this over ϕ, we find the charge that flows across circle of radius r:

$$\Delta Q = \sigma \Delta\Phi \int d\phi \, \frac{r}{2\pi rc} = \frac{\sigma\Phi_0}{c}.$$

   We learned on the previous problem set that $R_H = \sigma_H^{-1}$, so using $\sigma_H = e^2/3h$, we find

$$Q = \frac{e}{3}.$$

   The fractional charge of the excitation leads to a lot of very interesting physics; one notable effect is that the quasi-particles behave neither as fermions or as bosons, but rather as something in between ("anyons").

3. **Perturbation of the three-dimensional harmonic oscillator (13 points)**

   (a) **(2 points)** Creation and annihilation operators are given by, $a_i^\dagger$ and a_i respectively. Our system consists of three independent harmonic oscillators, so $E|n_1, n_2, n_3\rangle = \hbar\omega(n_1 + n_2 +$

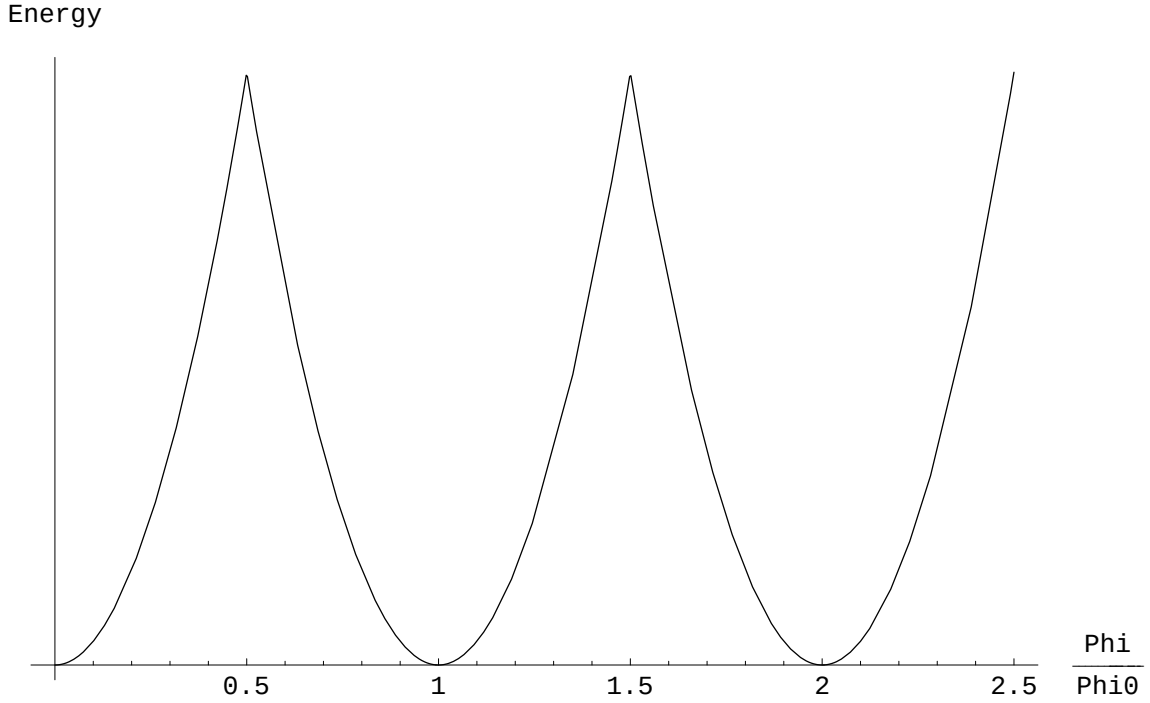

Figure 2: The lowest energy eigenvalue.

$n_3 + \frac{3}{2})|n_1, n_2, n_3\rangle$. Number operators are given by, $N_i = a_i^\dagger a_i$. If $E = \frac{5}{2}\hbar\omega$, then $\sum N_i = 1$, so there are three possible states: $|1,0,0\rangle$, $|0,1,0\rangle$, $|0,0,1\rangle$. We will take these states to be our basis (i.e., in this order).

(b) **(2 points)** Using $x_i = \sqrt{\frac{\hbar}{2m\omega}}(a_i + a_i^\dagger)$ and $p_i = i\sqrt{\frac{\hbar m\omega}{2}}(a_i^\dagger - a_i)$,

$$\begin{aligned} H_1 &= K\frac{i\hbar}{2}\left[(a_3 + a_3^\dagger)(a_1^\dagger - a_1) - (a_1 + a_1^\dagger)(a_3^\dagger - a_3)\right] \\ &= -iK\hbar(a_3^\dagger a_1 - a_1^\dagger a_3). \end{aligned}$$

(c) **(2 points)** The perturbation H_1 maps states 1 and 3 into each other, and annihilates state 2, so its matrix representation is given by

$$H_1 = iK\hbar \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

(d) **(3 points)** We can see immediately from part (c) that $|0,1,0\rangle$ is an eigenvector of H_1 with eigenvalue 0. We now only need to diagonalize $iK\hbar \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The eigenvalues of this two by two matrix are $\pm K\hbar$, with corresponding eigenvectors $\frac{1}{\sqrt{2}}(|1,0,0\rangle \mp i|0,0,1\rangle)$. All of these eigenvectors are also eigenvectors of H_0, so the total eigen-energies are $\frac{5}{2}\hbar\omega + K\hbar$, $\frac{5}{2}\hbar\omega - K\hbar$,

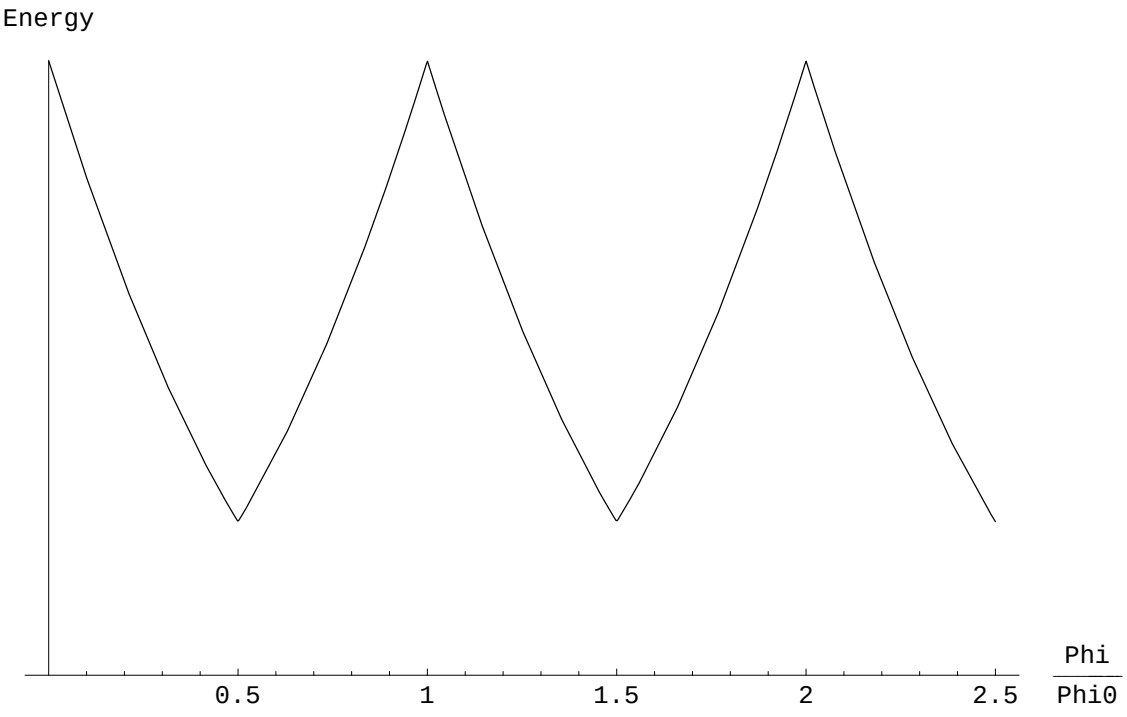

Figure 3: The second-lowest energy eigenvalue.

and $\frac{5}{2}\hbar\omega$. These energies are all distinct, so the degeneracy has been completely lifted by the perturbation.

(e) **(2 points)** Using the eigenvectors found in (d) as a basis, the total Hamiltonian has the following matrix representation in the degenerate subspace that we are studying:

$$H_0 + H_1 = \hbar \begin{pmatrix} \frac{5}{2}\omega + K & 0 & 0 \\ 0 & \frac{5}{2}\omega - K & 0 \\ 0 & 0 & \frac{5}{2}\omega \end{pmatrix}.$$

(f) **(2 points)** Since H_1 does not change the total number of excitations, $H_0 H_1 |\psi\rangle = H_1 H_0 |\psi\rangle$. In other words, $[H_0, H_1] = 0$, and therefore they are simultaneously diagonalizable (as we saw in the previous parts of the problem). The fact that H_0 and H_1 commute means that $\langle\phi|H_1|\psi\rangle = 0$, since we know $|\psi\rangle$ and $H_1|\phi\rangle$ are energy eigenstates with different energy eigenvalues.

4. **A delta-function interaction between two bosons in an infinite square well (8 points)**

In this problem we consider the potential

$$V(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq a, \\ \infty & \text{otherwise.} \end{cases}$$

The energy eigenvalues are

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2ma^2}, \tag{5}$$

and the corresponding wave functions are

$$\psi_n = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right). \tag{6}$$

(a) **(4 points)** The particles are independent, so to find the two particle wave functions we simply multiply the free wave functions given in (6) and symmetrize:

$$\begin{aligned}
\psi_{\text{ground}} &= \psi_1(x_1)\psi_1(x_2) = \frac{2}{a}\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right) \\
\psi_{\text{first}} &= \frac{1}{\sqrt{2}}\left[\psi_1(x_1)\psi_2(x_2) + \psi_2(x_1)\psi_1(x_2)\right] \\
&= \frac{\sqrt{2}}{a}\left[\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) + \sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right].
\end{aligned}$$

The energies of these wave functions are the sums of the free particle energies given in (5),

$$E_{\text{ground}} = \frac{\hbar^2\pi^2}{ma^2} \tag{7}$$

$$E_{\text{first}} = \frac{5\hbar^2\pi^2}{2ma^2}. \tag{8}$$

(b) **(4 points)** The first order correction to the ground state energy is

$$\begin{aligned}
\Delta E^{(1)}_{\text{ground}} &= -aV_0\int_0^a dx_1 dx_2 |\psi_1(x_1)|^2|\psi_1(x_2)|^2\delta(x_1-x_2) \\
&= -aV_0\left(\frac{2}{a}\right)^2\int_0^a dx_1\, \sin^4\left(\frac{\pi x_1}{a}\right) \\
&= -\frac{3V_0}{2}.
\end{aligned}$$

The first order correction to the first excited state energy is

$$\begin{aligned}
\Delta E^{(1)}_{\text{first}} &= -aV_0\int_0^a dx_1 dx_2 \left|\frac{1}{\sqrt{2}}(\psi_1(x_1)\psi_2(x_2) + \psi_2(x_1)\psi_1(x_2))\right|^2\delta(x_1-x_2) \\
&= -2aV_0\left(\frac{2}{a}\right)^2\int_0^a dx\, \sin^2\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{2\pi x}{a}\right) \\
&= -aV_0\frac{16}{a^2}\int_0^a dx\, \cos^2\left(\frac{\pi x}{a}\right)\sin^4\left(\frac{\pi x}{a}\right) \\
&= -2V_0.
\end{aligned}$$

Note that the shifts are independant of a.

5. **Anharmonic oscillator (14 points)**

(a) **(3 points)** The ground state wave function is even, while the perturbation λx^3 is odd. Therefore the integral $\int dx\, |\psi_0|^2\lambda x^3$ vanishes, and the first order contribution to the ground state energy is zero.

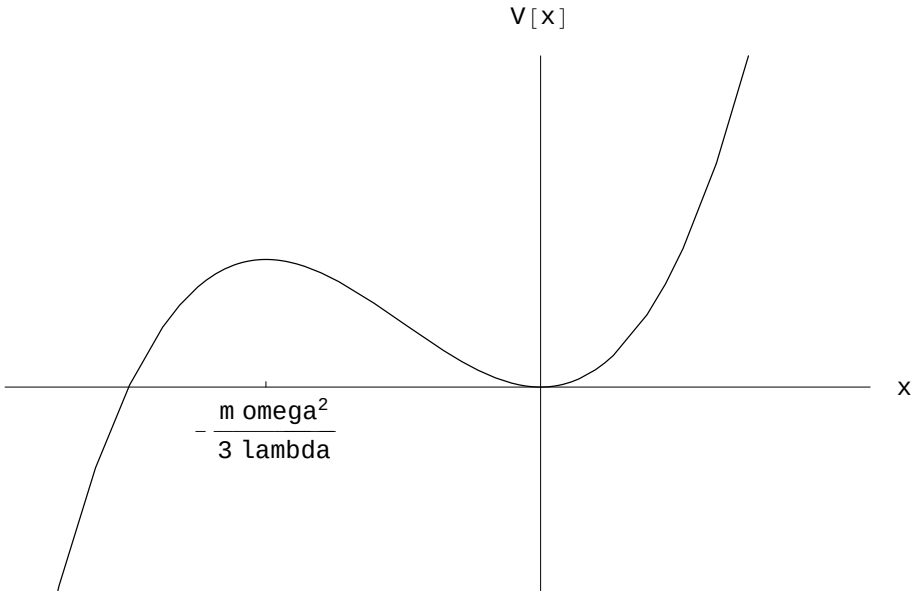

Figure 4: The potential λx^3.

To calculate the second-order shift, we need the expectation value $\langle n|(a+a^\dagger)^3|0\rangle$. We calculate $(a+a^\dagger)^3|0\rangle = 3|1\rangle + \sqrt{6}|3\rangle$, so the only contributions to the shift in the ground state energy will come from the states $|1\rangle$ and $|3\rangle$. We find, therefore,

$$\Delta E_0^{(2)} = \lambda^2 \left(\frac{\hbar}{2m\omega}\right)^3 \sum_{k>0} \frac{|\langle k|(a+a^\dagger)^3|0\rangle|^2}{E_0 - E_k} = -\lambda^2 \left(\frac{\hbar}{2m\omega}\right)^3 \left[\frac{9}{\hbar\omega} + \frac{(\sqrt{6})^2}{3\hbar\omega}\right],$$

which gives $\Delta E_0^{(2)} = -\frac{11\lambda^2\hbar^2}{8m^3\omega^4}$.

(b) **(3 points)** The first order correction to the ground state wave function is

$$\psi_0^{(1)} = \sum_m \frac{\langle m|H'|0\rangle}{E_0 - E_m}|m\rangle.$$

As in part (a), the only contributions are from $m = 3$ and $m = 1$:

$$\psi_0^{(1)} = -\frac{\lambda}{\hbar\omega}\left(\frac{\hbar}{2m\omega}\right)^{3/2}\left(3|1\rangle + \sqrt{\frac{2}{3}}|3\rangle\right).$$

The ground state wave function is then, to first order, $\psi_0 = |0\rangle - \frac{\lambda}{\hbar\omega}\left(\frac{\hbar}{2m\omega}\right)^{3/2}\left(3|1\rangle + \sqrt{\frac{2}{3}}\,|3\rangle\right)$.

(c) **(3 points)** A sketch of the potential is in figure 4. This potential is unbounded from below, so there is no ground state – any state localized near $x = 0$ is unstable, as it will eventually tunnel through the barrier. Perturbation theory does not account for the tunneling effects. It is good for examining relatively small, localized changes in the potential, but not for the cases like this, where the perturbation drastically changes the asymptotic behavior of the potential.

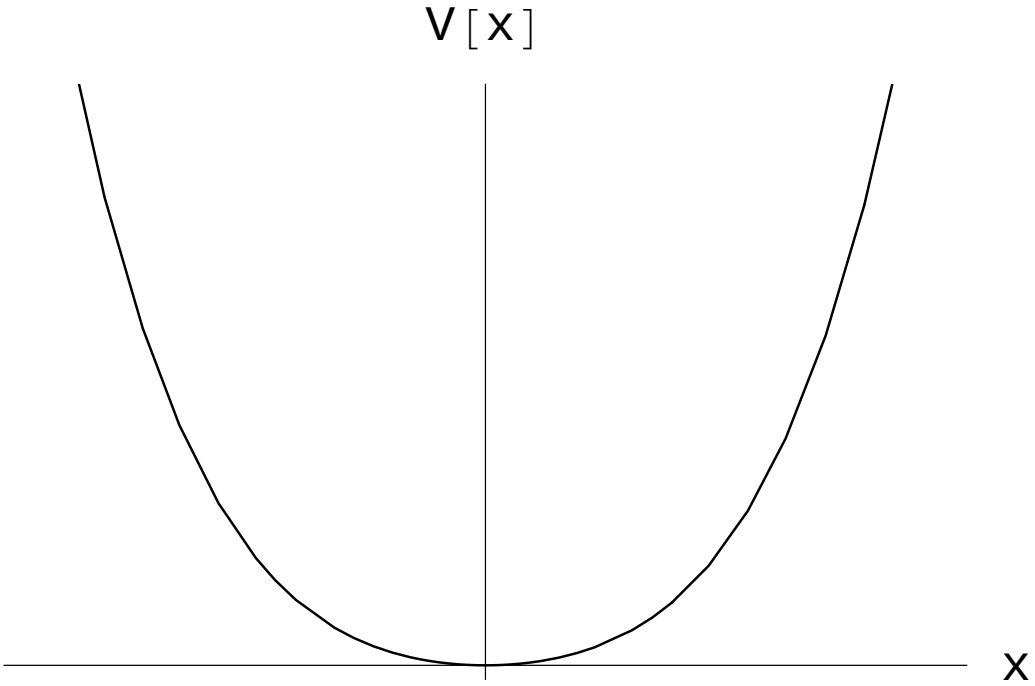

Figure 5: The potential $\frac{1}{2}m\omega^2x^2 + \lambda x^4$, with $\lambda > 0$.

The above qualitative argument is enough for full credit. However, it is nice to see quantitatively what is going on. You learned in 8.04 how to compute the probability of tunneling through a barrier. If you apply these methods to our problem, you will find that the probability for the particle to tunnel from $x = 0$ to $x = -\frac{m\omega^2}{2\lambda}$ goes like $\exp(-\text{const}/\lambda^2)$. This is nonzero, but we cannot expand it as a Taylor series in λ. Thus, to any finite order in perturbation theory, the probability of tunneling is zero.

(d) **(5 points)** We now take the perturbation to be $H' = \lambda x^4$. To calculate the first order shift in the ground state energy, we need $(a + a^\dagger)^4|0\rangle = 3|0\rangle + 6\sqrt{2}|2\rangle + 2\sqrt{6}|4\rangle$. We find that the first order shift in the ground state energy is

$$\Delta E_0^{(1)} = \langle 0|H'|0\rangle = \lambda \left(\frac{\hbar}{2m\omega}\right)^2 \langle 0|(a + a^\dagger)^4|0\rangle = 3\lambda \left(\frac{\hbar}{2m\omega}\right)^2 .$$

We sketch the behavior of the potential for positive and negative λ in figure 5.

If $\lambda > 0$, perturbation theory is a good approximation to the ground state, which "sees" a region of the potential where $\lambda x^4 \ll \frac{m\omega^2}{2}x^2$. (Perturbation theory will not work so well for higher states.) The contribution to the ground state energy at first order is positive, which makes physical sense – we are confining the particles with a sharper potential, which suggests that their energy should increase.

If, of the other hand, $\lambda < 0$, perturbation theory clearly fails. The change to the ground state energy is negative, so perturbation theory does see that the overall energy will be lowered.

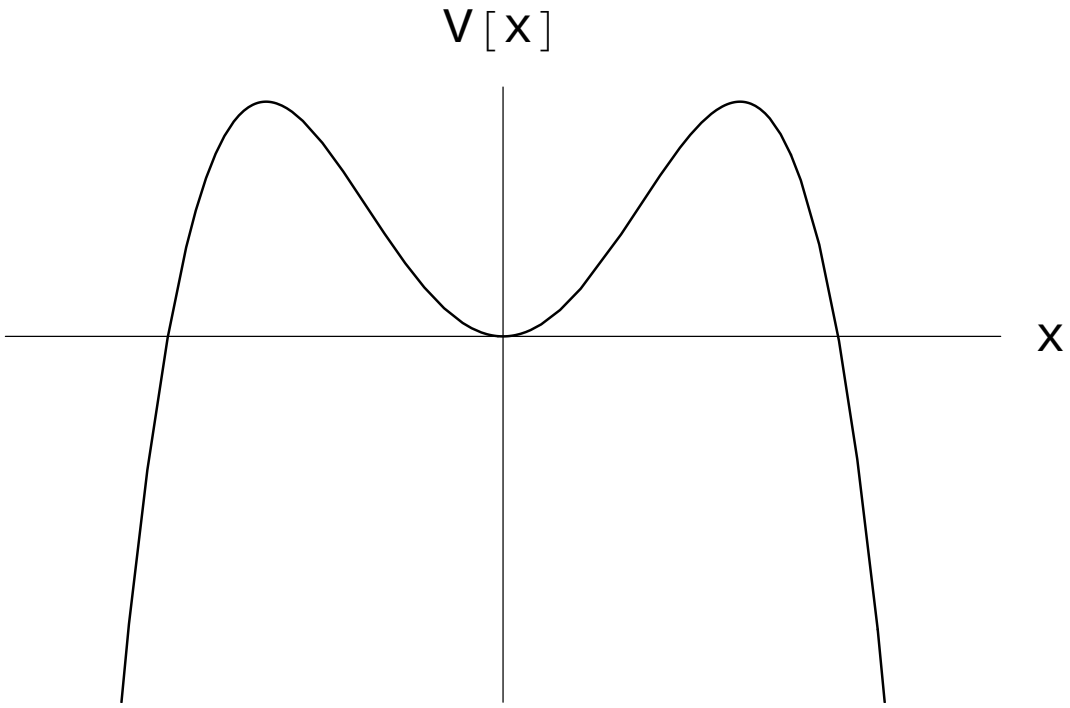

Figure 6: The potential $\frac{1}{2}m\omega^2x^2 + \lambda x^4$, with $\lambda < 0$.

However, as in part (c), perturbation theory doesn't "know" about the change in the asymptotic behavior of the potential, and cannot inform us about the tunneling that will result.

In cases like this, where flipping the sign of the perturbation parameter alters the asymptotic behavior of the perturbation, strictly speaking, the radius of convergence of perturbation theory as a series in λ is *zero*. Therefore, even when $\lambda > 0$, perturbation theory is an asymptotic expansion: it cannot capture all the physics, although it can still give us good information about the first few terms in the series.