Quantum Physics III (8.06) Spring 2007
Assignment 3

Feb 20, 2007 **Due Feb 27, 2007, 7pm**

- Please remember to put **your name and section time** at the top of your paper.

Readings

The current reading assignment is:

- Supplementary notes on Canonical Quantization and Application to a Charged Particle in a Magnetic Field.
- Griffiths Section 10.2.4 is an excellent treatment of the Aharonov-Bohm effect, but ignore the connection to Berry's phase for now. We will come back to this later.
- Cohen-Tannoudji Ch. VI Complement E
- Those of you reading Sakurai should read pp. 130-139.

Problem Set 3

Notes:

1. In this problem set, the following definitions are used throughout:

$$\omega_L \equiv \frac{qB}{mc}, \qquad l_0 \equiv \sqrt{\frac{\hbar}{m\omega_L}} = \sqrt{\frac{\hbar c}{qB}} \tag{1}$$

$$v_x = \frac{1}{m}\left(p_x - \frac{q}{c}A_x\right), \qquad v_y = \frac{1}{m}\left(p_y - \frac{q}{c}A_y\right) \tag{2}$$

2. Problems in this set are designed to help you understand better various aspects of Landau levels. Make sure to ask yourself what you have learned from each problem after you finish them.

1. **Gauge Invariance and the Schrödinger Equation (14 points)**

   The time dependent Schrödinger equation for a particle in a electromagnetic field given by

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A}(\vec{x},t)\right)^2\psi(\vec{x},t) + q\phi(\vec{x},t)\psi(\vec{x},t)\ . \tag{3}$$

   (a) (4 points) Consider a gauge transformation

$$\begin{aligned}\vec{A}'(\vec{x},t) &= \vec{A}(\vec{x},t) - \vec{\nabla}f(\vec{x},t)\\ \phi'(\vec{x},t) &= \phi(\vec{x},t) + \frac{1}{c}\frac{\partial f}{\partial t}(\vec{x},t)\ .\end{aligned} \tag{4}$$

   $(\vec{A}', \phi')$ and $(\vec{A}, \phi)$ describe the same $\vec{E}$ and $\vec{B}$. Show that if $\psi(\vec{x},t)$ solves the Schrödinger equation with $\vec{A}, \phi$ (which we will call "unprimed gauge"), then

$$\psi'(\vec{x},t) \equiv \exp\left(-\frac{iq}{\hbar c}f(\vec{x},t)\right)\psi(\vec{x},t) \tag{5}$$

   solves the Schrödinger equation with $\vec{A}', \phi'$ (which we will call "primed gauge").

   (b) (10 points) Physical observables should be gauge invariant. Check whether the following quantities are gauge invariant or not:

      i. $\langle\psi|\hat{x}_i|\psi\rangle$
      ii. $\langle\psi|\hat{p}_i|\psi\rangle$
      iii. $\langle\psi|\hat{v}_i|\psi\rangle$, where $\hat{v}_i = \frac{1}{m}\left(\hat{p}_i - \frac{q}{c}A_i\right)$
      iv. Energy expectation values (assuming that f is time-independent)
      v. Energy eigenvalues (assuming that f is time-independent)

   Gauge invariant operators are those whose expectation values in any states are gauge invariant.

2. **General aspects of quantum motion in a magnetic field (15 points)**

   The quantum motion for a particle in a magnetic field shows some resemblances to the classical motion and also many important differences. The differences can be traced to various commutators derived in this problem, in particular equations (6) and (9).

   The questions in this problem should be derived without explicitly choosing a gauge.

(a) (4 points) In this part we consider an arbitrary magnetic field (not necessarily constant). Find the commutator

$$[\hat{v}_x, \hat{v}_y] = ? \tag{6}$$

where $\hat{v}_{x,y}$ are the velocity operator defined by (2). What can you conclude about the motion of the particle from (6)?

(b) (4 points) In this and all parts below, we take $\vec{E} = 0$ and

$$B_x = B_y = 0, \qquad B_z = B = \text{const}$$

and look at the motion in $x - y$ plane only. Classically the particle travels in a circle around a "center of orbit" (X, Y), which can be expressed in terms of the coordinates (x, y) and the velocities (v_x, v_y) of the particle as (prob. 3 of pset 2)

$$X = x + \frac{v_y}{\omega_L}, \qquad Y = y - \frac{v_x}{\omega_L} \ . \tag{7}$$

Motivated by the classical expressions (7) we introduce quantum operators

$$\hat{X} = \hat{x} + \frac{\hat{v}_y}{\omega_L}, \qquad \hat{Y} = \hat{y} - \frac{\hat{v}_x}{\omega_L} \tag{8}$$

Find the commutators

$$[\hat{X}, \hat{Y}] = ? \tag{9}$$

What can you say about the motion in $x - y$ plane from (9)?

(c) (1 point) Show that $\hat{X}$ and $\hat{Y}$ are gauge invariant (see Prob. 1(b)).

(d) (6 points) Show that

$$[\hat{X}, H] = [\hat{Y}, H] = 0 \ . \tag{10}$$

Equations (9) and (10) imply that one of $\hat{X}$ and $\hat{Y}$ (or an arbitrary linear combination of them, but not both) can be diagonalized together with the Hamiltonian.

[Hint: It is convenient to write the Hamiltonian in a form $H = \frac{1}{2}m(\hat{v}_x^2 + \hat{v}_y^2)$ and first find the commutators between $\hat{X}, \hat{Y}$ and $\hat{v}_x, \hat{v}_y$.]

3. **Landau Levels: numerics (4 points)**

Consider a charged particle of charge q and mass m in a constant magnetic field B. The energy eigenvalues are separated by a spacing $\hbar\omega_L$ and typical wave functions are characterized by the length l_0 (see equation (1) for notations).

Suppose B is a field of 10 Tesla and $q = e$ (electron charge). (This is a very strong magnetic field but is certainly one which can be created in the laboratory.) In a 10 Tesla magnetic field, what is $\hbar\omega_L$ in eV? What is ℓ_0 in cm?

Useful facts: 1 Tesla $= 10^4$ gauss. The gauss is the cgs unit of B. This turns out to mean that if B is 1 gauss, then the force eB is 300 eV/cm. Also, $\hbar c = 197 \times 10^{-7}$eV cm. And, the mass of the electron is $m = 0.511\text{MeV}/c^2$.

4. **Transformation between basis vectors of different gauges (15 points)**

Consider a particle of charge q and mass m moving in a constant magnetic field $\vec{B} = (0, 0, B)$. In the gauge

$$A_x = -By, \qquad A_y = A_z = 0 \tag{11}$$

we found in lecture that a basis of energy eigenvectors in the lowest Landau level is given by (see equation (1) for notations)

$$\psi_0(x, y; k_x) = e^{ik_x x}\phi_0(y - y_0) \tag{12}$$

where

$$\phi_0(y - y_0) = \frac{1}{(\pi l_0^2)^{\frac{1}{4}}} e^{-\frac{(y-y_0)^2}{2l_0^2}} \tag{13}$$

is the ground state wave function for a harmonic oscillator of frequency ω_L and

$$y_0 = -\frac{c\hbar k_x}{qB} = -l_0^2 k_x \ .$$

(a) (2 points) Show that in gauge (11), (12) is an eigenvector of $\hat{Y}$ introduced in (8). Find the eigenvalue. Thus (12) diagonalize H and Y simultaneously.

(b) (4 points) Without doing any calculation, write down energy eigenvectors $\psi_0'(x, y; k_y)$ in the lowest Landau level for gauge choice

$$A_y' = Bx, \qquad A_x' = A_z' = 0 \ . \tag{14}$$

Show that $\psi_0'(x, y; k_y)$ are eigenvectors of $\hat{X}$ introduced in (8) and thus diagonalize H and $\hat{X}$ simultaneously. [Note: in contrast to (12), $\psi_0'(x, y; k_y)$ is a plane wave in y-direction while a harmonic oscillator in x-direction.]

(c) (2 points) Find the gauge transformation between (14) and (11).

(d) (7 points) From (9), discuss how $\psi'(x, y; k_y)$ of (b) and (12) should be related to each other. Check that $\psi'(x, y; k_y)$ can be written in terms of linear superpositions of (12) as

$$\psi_0'(x, y; k_y) = e^{-\frac{iq}{\hbar c}f} \frac{l_0}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk_x \, e^{-il_0^2 k_x k_y} \, \psi_0(x, y; k_x) \tag{15}$$

where f is the gauge transformation you find from (c). The factor $e^{-\frac{iq}{\hbar c}f}$ in (15) comes from the gauge transformation. [Hint: To prove (15), it is convenient to write equation (13) in terms of its Fourier transform

$$\phi_0(y-y_0) = \frac{l_0}{(\pi l_0^2)^{\frac{1}{4}}} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, e^{ik(y-y_0)-\frac{l_0^2k^2}{2}}$$

and use it in the right hand side of (15).]

[**Note:** The fact that energy eigenvectors look so different in different gauges is due to the infinite degeneracy of each Landau level. Different choices of gauges select different sets of basis.]

5. **Electromagnetic Current Density in Quantum Mechanics (12 points)**

The probability flux in the Schrödinger equation can be identified as the electromagnetic current density, provided the proper attention is paid to the effects of the vector potential. This current density will play a role in our discussion of the quantum Hall effect.

Way back in the 8.04 you derived the probability flux in quantum mechanics:

$$\vec{S}(\vec{x},t) = \frac{\hbar}{m}\mathrm{Im}\left[\psi^*\vec{\nabla}\psi\right] \ . \tag{16}$$

In the presence of electric and magnetic fields, the probability current is modified to

$$S_i(\vec{x},t) = \frac{\hbar}{m}\mathrm{Im}\left[\psi^*\partial_i\psi\right] - \frac{q}{mc}\psi^*\psi A_i = \mathrm{Re}\left(\psi^*\hat{v}_i\psi\right) \tag{17}$$

This probability flux is conserved and when multiplied by q, the particle's charge, it can be interpreted as the electromagnetic current density, $\vec{j} \equiv q\vec{S}$.

(a) (7 points) Derive the expression eq. (17) for the probability flux. [Hint: It is convenient to work in a gauge where $\vec{\nabla}\cdot\vec{A} = 0$. The derivation of eq. (17) is parallel to that of (16), i.e. you need to derive

$$\frac{\partial\rho}{\partial t} = -\vec{\nabla}\cdot\vec{S}$$

with $\rho = \psi^*\psi$ and $\vec{S}$ given by eq. (17).]

(b) (2 points) Assuming that ψ has units $1/l^{3/2}$ as one would expect from the normalization condition, $\int d^3x\psi^*\psi = 1$, show that $\vec{j} = q\vec{S}$ has units of charge per unit area per unit time, which are the dimensions of current density.

(c) (3 points) Now show that $\vec{S}$ has *exactly the same form* in any gauge, that is, show that under gauge transformations (4) and (5), $\vec{S}'$ defined in terms of $\vec{A}'$ and ψ' is identical to $\vec{S}$ defined in terms of $\vec{A}$ and ψ.