Quantum Physics III (8.06) Spring 2008
Final Exam Solutions

May 19, 2008

1. **Short answer questions (35 points)**

   (a) (2 points) $\alpha^4 mc^2$

   (b) (2 points) $\mu_B B$, where $\mu_B = \frac{e\hbar}{2m}$

   (c) (3 points) In the variational ansatz, take each electron to be in the electric field of a nucleus with charge $Ze < 2e$.

   (d) (3 points) $\frac{e^2}{h}$

   (e) (3 points)

   $$\frac{d\sigma}{d\Omega} = \frac{\left(\frac{dN}{d\Omega dt}\right)_{\text{detector}}}{\left(\frac{dN}{dAdt}\right)_{in}} \tag{1}$$

   (f) The scattering boundary condition (2 points): as $r \to \infty$

   $$\psi \to e^{ikz} + \frac{e^{ikr}}{r} f(\theta, \phi) \tag{2}$$

   The differential cross section can be written as (2 points)

   $$\frac{d\sigma}{d\Omega} = |f(\theta, \phi)|^2 \tag{3}$$

   (g) (3 points) $\Phi = n\frac{hc}{e}$ with n an integer.

   (h) (3 points) $(10^{13})^{\sqrt{2}}$

   (i) (3 points) $(10^{13})^2$

   (j) (3 points) $(10^{13})^{\sqrt{2}}$

   (k) (6 points)

   $$1 \to 5, \quad 2 \to 6, \quad 3 \to 3, \quad 4 \to 4, \quad 5 \to 1, \quad 6 \to 2\ .$$

2. **Particle on a ring (10 points)**

   (a) For each eigenvalue, it is two-fold degenerate (2 points). The WKB wave functions are (3 points)

   $$\psi(x) \sim \frac{1}{\sqrt{p(x)}} e^{\pm \frac{i}{\hbar} \int_0^x dx' p(x')} \tag{4}$$

   where

   $$p(x) = \sqrt{2m(E - V(x))} \tag{5}$$

(b) (5 points) The wave functions should be single-valued on the circle, i.e. $\psi(0) = \psi(L)$. Thus we require

$$\int_0^L dx\, p(x) = 2\pi n\hbar, \qquad n = \pm 1, \pm 2, \cdots \tag{6}$$

3. **Born approximation (16 points)**

(a) (10 points) The 1st Born approximation for the amplitude of scattering off a single atom at $\vec{x} = 0$ is given by

$$\begin{aligned} f_1(\theta,\phi) &= -\frac{m}{2\pi\hbar^2}\int d^3x\, \exp(-i\vec{q}\cdot\vec{x})v(\vec{x}) \\ &= -\frac{mv_0}{2\pi\hbar^2}\int d^3x\, \exp\left(-i\vec{q}\cdot\vec{x} - \frac{\vec{x}^2}{2b^2}\right) \\ &= -\frac{\sqrt{2\pi}b^3 mv_0}{\hbar^2} e^{-\frac{1}{2}b^2\vec{q}^2} \end{aligned} \tag{7}$$

The simplest way to evaluate the integral in the second line is to use the Cartesian coordinates and it then becomes a product of three Gaussian integrals.

Thus we find that

$$\frac{d\sigma_1}{d\Omega} = |f_1(\theta)|^2 = \left(\frac{\sqrt{2\pi}b^3 mv_0}{\hbar^2}\right)^2 e^{-b^2\vec{q}^2}$$

with

$$q^2 = 4k^2 \sin^2\frac{\theta}{2}$$

(b) (6 points) Now with the full potential, we find that

$$\begin{aligned} f(\theta,\phi) &= = -\frac{m}{2\pi\hbar^2}\int d^3x\, \exp(-i\vec{q}\cdot\vec{x})V(\vec{x}) \\ &= -\frac{mv_0}{2\pi\hbar^2}\sum_i \int d^3x\, \exp(-i\vec{q}\cdot\vec{x})v(\vec{x} - \vec{X}_i) \\ &= \left(\sum_i e^{-i\vec{q}\cdot\vec{X}_i}\right) f_1(\theta,\phi) \end{aligned} \tag{8}$$

We thus find that

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma_1}{d\Omega}\left|\sum_i e^{-i\vec{q}\cdot\vec{X}_i}\right|^2 \tag{9}$$

4. **Partial wave expansion (20 points)**

(a) (2 point) Since the first term in $f(\theta)$ is independent of θ, and the second term is proportional to $\cos\theta$, the partial waves that are active are $\ell = 0$ and $\ell = 1$.

(b) (10 points) Recalling the expansion $f(\theta) = \sum_\ell (2\ell+1)P_\ell(\cos\theta)f_\ell$, we identify (2 points)

$$\begin{aligned} f_0 &= \frac{1}{k}\frac{\Gamma k}{k_0 - k - ik\Gamma} \\ &= \frac{1}{k}\frac{\Gamma k(k_0 - k + ik\Gamma)}{(k_0-k)^2 + (\Gamma k)^2}, \end{aligned}$$

Since $f_0 = \frac{1}{k} e^{i\delta_0} \sin\delta_0 = \frac{1}{k}(\cos\delta_0 \sin\delta_0 + i \sin^2\delta_0)$, we find that (3 points)

$$\sin\delta_0 = \frac{\Gamma k}{\sqrt{(k_0 - k)^2 + (\Gamma k)^2}}.$$

As $k \to 0$, $\sin\delta_0 \simeq \delta_0 \simeq (\Gamma/k_0)k$. This goes like k, as we expect (1 point).
For $\ell = 1$, we have (2 points)

$$\begin{aligned} f_1 &= \frac{1}{k} e^{2i\beta k^3} \sin(2\beta k^3) \\ &= \frac{1}{k} \sin\delta_1 e^{i\delta_1}, \end{aligned}$$

and therefore (1 point)

$$\delta_1 = 2\beta k^3.$$

This goes to 0 as k^3, again as we expect (1 point).

(c) (3 point) the total cross-section is

$$\sigma_{tot} = \frac{4\pi}{k^2}\left(\sin^2\delta_0 + 3\sin^2\delta_1\right) = \frac{4\pi}{k^2}\left[\frac{(\Gamma k)^2}{(k_0 - k)^2 + (\Gamma k)^2} + 3\sin^2(2\beta k^3)\right] \tag{10}$$

(d) (5 points) The optical theorem is (3 points)

$$\sigma_{tot} = \frac{4\pi}{k}\mathrm{Im} f(\theta = 0) \tag{11}$$

Since (2 points)

$$\mathrm{Im} f(0) = \frac{1}{k}\left[\frac{(\Gamma k)^2}{(k_0 - k)^2 + (\Gamma k)^2} + 3\sin^2(2\beta k^3)\right].$$

we conclude that the optical theorem is satisfied.

5. **Semi-classical approximation of phase shift (20 points)**

(a) (5 points) As $r \to 0$, $u(r) \to 0$ (2 points). As $r \to \infty$ (3 points),

$$u(r) \sim e^{-ikr} - e^{2i\delta_0} e^{ikr} \sim e^{i\delta_0} \sin(kr + \delta_0) \tag{12}$$

(b) (5 points) The WKB wave function can be written as

$$\psi(r) = \frac{c_+}{\sqrt{p(r)}} e^{i\int_0^r dx\, p(x)} + \frac{c_-}{\sqrt{p(r)}} e^{-i\int_0^r dx\, p(x)} \tag{13}$$

with

$$p(r) = \sqrt{k^2 - U(r)}, \qquad U(r) = \frac{2m}{\hbar^2} V(r)$$

Note that $p(r)$ defined above is related by standard definition by a factor of $1/\hbar$.
The boundary condition at $r = 0$ implies that

$$\psi(r) \propto \frac{1}{\sqrt{p(r)}} \sin\left(\int_0^r dx\, p(x)\right) \tag{14}$$

(c) (6 points) Note that as $r \to \infty$,

$$\int_0^r dx\, p(x) \to kr + \int_0^\infty dx\, (p(x) - k) + O(1/r) \tag{15}$$

Comparing equations (12), (14) and (15) we thus conclude that

$$\delta_0 = \int_0^\infty dx\, (p(x) - k) = \int_0^\infty dx\, \left(\sqrt{k^2 - U(x)} - k\right) \tag{16}$$

(d) (4 points) As $k \to 0$, we find that (note that $U(r) < 0$) (2 points)

$$\delta_0 = \int_0^\infty dx\, \sqrt{-U(r)} \tag{17}$$

which is not the right behavior, since we expect $\delta_0 \propto k$ as $k \to 0$. The reason is that for $k \to 0$, E can get close to $V(r)$ and thus the WKB approximation is not good. (2 points)

6. **A time-dependent two-state system (16 points)**

(a) (4 points) We should choose $A(t)$ so that $|A(\pm\infty)| \gg |B|$, while $A(\infty)$ and $A(-\infty)$ have opposite sign (3 points). An example is $A(t) = vt$ with v a small, positive constant.

(b) (4 points) The gap at $t = 0$ is given by $2B$. The time period δt in which the gap between two states of system is of order B is

$$|v\delta t| \sim B \quad \to \quad |\delta t| \sim \frac{B}{v}$$

We need

$$\frac{\hbar}{\delta t} \ll 2B \quad \to \quad \frac{B^2}{\hbar v} \gg 1$$

(c) (8 points) Note that the eigenvalues of $H(t)$ are time-dependent and are given by $\pm K$. At $t = 0$ the system is in an eigenvector of $H(t = 0)$ with eigenvalue K. In adiabatic approximation, the system will remain in the eigenvector of $H(t)$ with eigenvalue K. At time $T = \frac{2\pi}{f}$, where we have $H(T) = H(0)$, we thus have (2 points)

$$\psi(T) = e^{-i\theta_+(T) + i\gamma_+(T)} \psi(t = 0) \tag{18}$$

Thus we find the dynamical phase is given by (2 points)

$$\theta_+(T) = \frac{1}{\hbar} KT = \frac{2\pi K}{\hbar f}$$

To find the geometric phase, we note that the Hamiltonian is the same as that of a spin-$\frac{1}{2}$ particle in a rotating magnetic field given by (2 points)

$$\vec{B} \propto (K \sin ft, 0, K \cos ft)$$

At from $t = 0$ to $t = T = 2\pi f$, the magnetic field transverses a full circle in the $x - z$ plane. The geometry phase is then given by (since the solid angle of the circle with respect to the origin is 2π) (2 points)

$$\gamma_+(T) = -\pi$$

Thus we find that the total phase factor

$$-e^{-i\frac{2\pi K}{\hbar f}}$$

7. **Two-dimensional anharmonic oscillator: perturbation theory (24 points)**

In the problem we set $m = \omega = \hbar = 1$.

Note that

$$x_1 = \frac{1}{\sqrt{2}}(a_1 + a_1^\dagger), \qquad x_2 = \frac{1}{\sqrt{2}}(a_2 + a_2^\dagger)$$

Thus

$$H' = \frac{1}{2}\lambda(a_1 + a_1^\dagger)^2(a_2 + a_2^\dagger)^2$$

(a) (4 points) Θ commutes with the full Hamiltonian since the Hamiltonian is invariant under the exchange of labels $1 \leftrightarrow 2$ (2 points). Thus one can diagonalize H and Θ simultaneously and the energy eigenfunction should satisfy (2 points)

$$\psi_n(x_1, x_2) = \pm\psi_n(x_2, x_1) \tag{19}$$

(b) (6 points) The ground state is $|0,0\rangle$. So we have

$$\delta E_{00} = \frac{1}{2}\lambda\langle 0,0|(a_1 + a_1^\dagger)^2(a_2 + a_2^\dagger)^2|0,0\rangle = \frac{1}{2}\lambda\langle 0,0|a_1 a_1^\dagger a_2 a_2^\dagger|0,0\rangle = \frac{1}{2}\lambda$$

(c) (7 points) The first order correction to the ground state wave function can be written as

$$\psi_{00}^{(1)} = \sum_{(n,m)\neq(0,0)} |m,n\rangle \frac{\langle m,n|H'|00\rangle}{E_{00} - E_{m,n}} \tag{20}$$

Note that

$$\langle m,n|H'|0,0\rangle = 2\lambda\langle m|x_1^2|0\rangle\langle n|x_2^2|0\rangle$$

where $|n\rangle$ denote an energy eigenstates for a single harmonic oscillator. Since

$$\langle m|x^2|0\rangle = \frac{1}{2}\langle m|(a + a^\dagger)^2|0\rangle = \frac{\sqrt{2}}{2}\delta_{m,2} + \frac{1}{2}\delta_{m,0}$$

we conclude that the only nonzero matrix elements in $\langle m,n|H'|0,0\rangle$ for $(m,n) \neq (0,0)$ are

$$\langle 2,0|H'|0,0\rangle = \langle 0,2|H'|0,0\rangle = \frac{\sqrt{2}}{2}\lambda, \qquad \langle 2,2|H'|0,0\rangle = \lambda \tag{21}$$

where the second expression is obtained by $1 \leftrightarrow 2$ symmetry. Note that $E_{2,0} - E_{0,0} = E_{0,2} - E_{0,0} = 2$ and $E_{2,2} - E_{0,0} = 4$. We thus find that

$$\psi_{00}^{(1)} = -\frac{\lambda}{4}(\sqrt{2}|0,2\rangle + \sqrt{2}|2,0\rangle + |2,2\rangle) \tag{22}$$

This expression is symmetric under $1 \leftrightarrow 2$ as one would expect for the ground state.

(d) (7 points) There are two first excited states $|0,1\rangle$ and $|1,0\rangle$. We thus need to use degenerate perturbation theory. Since Θ commutes with H. The "good states" should be eigenstates of Θ, in which H is automatically diagonal. The good states are

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle \pm |0,1\rangle), \qquad \Theta|\pm\rangle = \pm|\pm\rangle$$

We thus need only to compute $\langle +|H'|+\rangle$ and $\langle -|H'|-\rangle$. For this purpose, let us note that

$$\langle 0,1|H'|0,1\rangle = \frac{1}{2}\lambda\langle 0,1|(a_1+a_1^\dagger)^2(a_2+a_2^\dagger)^2|0,1\rangle = \frac{3\lambda}{2} = \langle 1,0|H'|1,0\rangle$$
$$\langle 0,1|H'|1,0\rangle = \frac{1}{2}\lambda\langle 0,1|(a_1+a_1^\dagger)^2(a_2+a_2^\dagger)^2|1,0\rangle = 0$$

Since the off-diagonal terms are zero, we find that the first excited states remind degenerate to 1st order with an energy correction $\delta E = \frac{3\lambda}{2}$. In particular the Hamiltonian is diagonal in any choice of basis. (Due to this degeneracy the basis of "good states" does not have a special role at this order.)

8. **Two-dimensional anharmonic oscillator: variational approach (23 points)**

Note: in choosing trial wave functions, we should consider functions which have right symmetry properties discussed in part (a) of last problem.

(a) (5 points) The trial wave function for the ground state should be symmetric under the exchange of 1 and 2. We thus consider the following (normalized) trial wave function

$$\psi(x_1,x_2) = \sqrt{\frac{a}{\pi}}e^{-\frac{1}{2}a(x_1^2+x_2^2)} \tag{23}$$

The unperturbed ground state wave function of H_0 corresponds to setting $a=1$ in the above equation. Thus if we minimize the energy expectation value of ψ with respect to a, we are warranted to do no worse than the first order perturbation theory.

(b) (12 points) We need to minimize

$$E(a) = \langle\psi|H_0+H'|\psi\rangle \tag{24}$$

Note that

$$\langle\psi|H'|\psi\rangle = 2\lambda\left(\frac{a}{\pi}\right)\int_{-\infty}^{\infty}dx_1dx_2\,x_1^2x_2^2\,e^{-a(x_1^2+x_2^2)} = \frac{\lambda}{2a^2}$$
$$\langle\psi|H_0|\psi\rangle = \left(\frac{a}{\pi}\right)(1+a^2)\int_{-\infty}^{\infty}dx_1dx_2\,x_1^2\,e^{-a(x_1^2+x_2^2)} = \frac{1+a^2}{2a}$$

Minimizing $E(a)$ we find that

$$-\frac{\lambda}{a^3}+\frac{1}{2}-\frac{1}{2a^2} = 0 \quad \rightarrow \quad a^3-2\lambda-a=0$$

In the large λ limit, we find that

$$a_{min} = (2\lambda)^{\frac{1}{3}}$$

and an upper bound on energy

$$E(a_{min}) = \frac{3}{4}a_{min}$$

(c) (6 points) For example, the trial functions for one of the first excited states can be chosen to be

$$\psi(x_1, x_2) = A(x_1 - x_2)e^{-\frac{a}{2}(x_1^2+x_2^2)}$$

This wave function is antisymmetric in $1 \leftrightarrow 2$ and includes $|1,0\rangle - |0,1\rangle$ as a special example. Thus it is orthogonal to the (exact) ground state and will give an estimate of energy no worse than the first order perturbation theory.

(Note in grading we gave full credits to any wave functions which are orthogonal to the ground state trial wave functions.)

9. **Three-dimensional harmonic oscillators: Transition amplitudes (16 points)**

(a) (6 points) Note the amplitude to be in a state $|b\rangle$ starting from a state $|a\rangle$ is given by (with $\hbar = m = \omega = 1$)

$$c_b = \frac{1}{i}\int_0^T dt\ H'_{ba}(t)e^{i\omega_{ba}t'}, \qquad \omega_{ba} = E_b - E_a$$

In our case

$$H'_{ba} = \lambda\langle 1,0,0|(a_1 + a_1^\dagger)(a_2 + a_2^\dagger)|0,1,0\rangle = \lambda$$

$$\omega_{ba} = 0$$

We thus find that

$$c_b = -i\lambda T$$

and

$$P_{a\to b} = \lambda^2 T^2$$

(b) (3 points) For perturbation theory to be valid we need the first order wave function to be much smaller than zeroth order one, i.e. $|c_b| \ll 1$, which leads

$$\lambda T \ll 1$$

Note that requiring $\lambda \ll 1$ is not enough.

(c) (7 points) Note that the rate for spontaneous emission is given by

$$A = \frac{4\omega_{ba}^3}{3c^3}|\vec{p}_{ab}|^2$$

where

$$\vec{p}_{ab} = \langle 0,0,0|e\vec{r}|1,0,0\rangle$$

is the matrix element of the dipole operation $e\vec{r}$.

The only nonzero component of the above expression is

$$p_{ab}^{(x)} = \frac{e}{\sqrt{2}}\langle 0,0,0|a_1 + a_1^\dagger|1,0,0\rangle = \frac{e}{\sqrt{2}}$$

while

$$\omega_{ba} = 1$$

We thus find that

$$A = \frac{2e^2}{3c^3}$$

The lifetime

$$\tau = \frac{1}{A} = \frac{3c^3}{2e^2}$$

Note that restoring m, ω and $\hbar$ the final expression becomes

$$A = \frac{2e^2\omega^2}{3mc^3} = \frac{2\omega}{3}\frac{e^2}{\hbar c}\frac{\hbar\omega}{mc^2}$$