Quantum Physics III (8.06) Spring 2008
MIDTERM TEST
Thursday March 20, 2008
You have 1 hour and 20 minutes.

There are 5 problems, totalling 80 points. Do all problems.

Answer all problems in the white books provided.
Write YOUR NAME on EACH white book you use.
Budget your time wisely, using the point values as a guide. Note that shorter problems may not always be easier problems.

No books, notes or calculators allowed.

Some potentially useful information

- Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle$$

For an energy eigenstate ψ of energy E

$$\psi(t) = e^{-\frac{i}{\hbar}Et}\psi(0)$$

and the Schrodinger equation reduces to an eigenvalue equation

$$H\psi = E\psi$$

- Harmonic Oscillator

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$$

where

$$[\hat{x}, \hat{p}] = i\hbar \ .$$

This Hamiltonian can be rewritten as

$$\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$$

where $\hat{N} = \hat{a}^\dagger\hat{a}$, and the operators $\hat{a}$ and $\hat{a}^\dagger$ are given by

$$\hat{a} = \frac{1}{\sqrt{2m\omega\hbar}}\left(m\omega\hat{x} + i\hat{p}\right) \ , \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2m\omega\hbar}}\left(m\omega\hat{x} - i\hat{p}\right) \ ,$$

and satisfy

$$[\hat{a}, \hat{a}^\dagger] = 1 \ .$$

Conversely

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger), \qquad \hat{p} = \frac{1}{i}\sqrt{\frac{\hbar m\omega}{2}}(\hat{a} - \hat{a}^\dagger)$$

The action of $\hat{a}$ and $\hat{a}^\dagger$ on eigenstates of $\hat{N}$ is given by

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \ , \quad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle \ .$$

The ground state wave function is

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right) \ .$$

- Useful integrals

$$\int_{-\infty}^{+\infty} dx \exp\left(-ax^2\right) = \sqrt{\frac{\pi}{a}}$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{in\phi}\cos m\phi = \frac{1}{2}(\delta_{m,n}+\delta_{m,-n})$$

- Particle in an Electric and/or Magnetic Field:

  The Hamiltonian for a particle with charge q in a magnetic field and electric field

$$\vec{B} = \vec{\nabla}\times\vec{A}, \qquad \vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

  is:

$$H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + q\phi \tag{1}$$

  Note:

$$B_z = \partial_x A_y - \partial_y A_x$$

- Gauge invariance:

  If $\psi(\vec{x},t)$ solves the Schrödinger equation defined by the Hamiltonian (1), then

$$\psi'(\vec{x},t) = \exp\left(-\frac{iq}{\hbar c} f(\vec{x},t)\right)\psi(\vec{x},t)$$

  solves the Schrödinger equation obtained upon replacing $\vec{A}$ by $\vec{A}' = \vec{A} - \vec{\nabla} f$ and replacing ϕ by $\phi' = \phi + (1/c)\partial f/\partial t$.

- Time independent perturbation theory:

  Suppose that

$$H = H_0 + H'$$

  where we already know the eigenvalues E_n^0 and eigenstates $|\psi_n^0\rangle$ of H^0:

$$H_0|\psi_n^{(0)}\rangle = E_n^{(0)}|\psi_n^{(0)}\rangle\ .$$

  Then, the eigenvalues and eigenstates of the full Hamiltonian H are:

$$E_n = E_n^{(0)} + H'_{nn} + \sum_{m\neq n}\frac{|H'_{nm}|^2}{E_n^{(0)} - E_m^{(0)}} + \ldots \tag{2}$$

$$|\psi_n\rangle = |\psi_n^{(0)}\rangle + \sum_{m\neq n}\frac{H'_{mn}}{E_n^{(0)} - E_m^{(0)}}|\psi_m^{(0)}\rangle + \ldots \tag{3}$$

  where $H'_{nm} \equiv \langle\psi_n^{(0)}|H'|\psi_m^{(0)}\rangle$.

  If H_0 has degeneracy at $E_n^{(0)}$, first diagonalize H' in the corresponding degenerate subspace, then use equations (2) and (3). In particular $|\psi_n^{(0)}\rangle$ ("good states") should be one of the eigenvectors of H' in the degenerate subspace.

1. **True or false (10 points)**

   (a) (2 points) After taking into account of the fine structure splittings, no states of different orbital angular momentum are any longer degenerate.

   (b) (2 points) Using the Aharonov-Bohm effect one can in principle measure the exact amount of magnetic flux inside an solenoid even though electrons move only in the region with no magnetic field.

   (c) (2 points) The ground state of a hydrogen atom does not have any permanent electric dipole moment.

   (d) (2 points) Consider a particle in a crossed electric and magnetic field. All states in the lowest Landau level are degenerate.

   (e) (2 points) In the approximation we used, the Chandrasekhar mass of a white dwarf star is not sensitive to the value of the electron mass. (For example, if the electron mass were half its observed value, the Chandrasekhar mass for a white dwarf star would still be the same.)

2. **Short answer questions (15 points)**

   (a) (3 points) Use one or two sentences to explain how a white dwarf star balances its gravitational attraction.

   (b) (3 points) What is the distinction between the strong-field and weak-field Zeeman effect?

   (c) (3 points) Give a brief proof that a perturbation of the Hamiltonian always lowers the energy of the ground state in **second order**.

   (d) (3 points) Consider a two-dimensional electron gas with total N electrons confined in a finite size sample with area A. Apply a magnetic field B perpendicular to the sample. Characterize the ground state of the system. (**You can ignore the spin of the electrons.**)

   (e) (3 points) What is the Bloch theorem?

3. **Two-dimensional free electron gas in a box (17 points)**

Consider a non-relativistic electron gas confined in a very thin sample of dimension $L \times L$. You can assume that the sample is sufficiently thin that the electrons can only move in the $x - y$ plane. The total number of electrons is N. Imagine that N is very large (say 10^{23}) so that you can ignore any boundary effects. Denote the mass of an electron by m.

(a) (3 points) Write down the single-particle energy eigenfunctions and eigenvalues. (You can assume that the wave functions vanish at the boundary of the sample.)

(b) (4 points) Find the location of the Fermi surface k_F.

(c) (6 points) What is the energy density (energy per unit area) and pressure (force per unit length) of the ground state?

(d) (4 points) Now consider instead N **bosons** confined in the same sample. Find the pressure for the ground state. Describe the qualitative difference between the pressure for bosons and that for fermions.

4. **Particle in a magnetic field (15 points)**

Consider a particle with charge q and mass m in a constant magnetic field field $\vec{B} = (0, 0, B)$. The particle is restricted to move in the (x, y) plane.

(a) (2 points) Choose a gauge so that the Hamiltonian commutes with the angular momentum operator L_z. (You can just directly write down the answer without verifying it.)

(b) (2 points) Find a gauge in which the Hamiltonian commutes with p_y. (You can just directly write down the answer without verifying it.)

(c) (6 points) Derive the eigenvalues of the Hamiltonian using the gauge of (b). (You should show intermediate steps. It is NOT enough just to write down the answer by memory.)

(d) (5 points) Find the expectation values of v_x and v_y for the particle in the lowest Landau level. (You should be able to find the answer without doing extensive calculations. If you can guess the answer, that is also fine. But you should clearly state your reasoning for the guess in order to get full credit.)

5. **Particle on a Sphere (23 points)**

Note: spherical harmonics and integrals that you might need for this problem are given at the end of the problem.

Consider a particle with mass M restricted to move on a sphere with radius $r = 1$. The wave function for the particle is thus a function only of the angles θ and ϕ. In the absence of any potential, the Hamiltonian for this system is

$$H_0 = \frac{L^2}{2M} \tag{4}$$

where L is the three-dimensional angular momentum operator.

The particle has charge q.

(a) (3 points) What are the energies and degeneracies of the states with angular momentum quantum number $l = 0, 1$?

(b) (4 points) Suppose the particle has spin 1/2 and consider the spin-orbit coupling of the form

$$H_1 = \frac{1}{M}\vec{L}\cdot\vec{S} \tag{5}$$

Find *exact* energy eigenvalues for $H = H_0 + H_1$ (you do not need to state the degeneracy).

For the following parts, ignore the spin of the particle and set $H_1 = 0$.

(c) (4 points) Now, turn on a magnetic field in the z-direction $\vec{B} = B_0 e_z$. To first order in B_0, the term added to the Hamiltonian as a result of the magnetic field is given by

$$H' = -\frac{q}{2M}\vec{B}\cdot\vec{L} \tag{6}$$

Calculate the energies of the $l = 1$ state(s) to first order in B_0.

(d) (6 points) Now turn off the magnetic field in (c) and consider a perturbation

$$H' = \epsilon\cos 2\phi\ . \tag{7}$$

Find the energy of $l = 1$ states to lowest order in ϵ.

(e) (6 points) Turn off the perturbation from part (d). Now, turn on an electric field in the z-direction, $\vec{E} = (0, 0, E_0)$. Calculate the energies of the $l = 0$ state(s) to second order in E_0.

Useful information for this problem:

(a) Spherical coordinates

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta \tag{8}$$

(b) Spherical harmonics

$$Y_{00} = \frac{1}{\sqrt{4\pi}}, \qquad Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta, \qquad Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi}, \qquad Y_{1-1} = \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\phi} \tag{9}$$

Orthogonality:

$$\int_0^\pi d\theta\, \sin\theta \int_0^{2\pi} d\phi\, Y^*_{lm}(\theta,\phi)\, Y_{l'm'}(\theta,\phi) = \delta_{ll'}\delta_{mm'} \tag{10}$$

(c) Useful integrals:

$$\int_0^\pi d\theta\, \sin\theta = 2, \qquad \int_0^\pi d\theta\, \sin\theta\cos\theta = 0, \qquad \int_0^\pi d\theta\, \sin^3\theta = \frac{4}{3} \tag{11}$$

$$\int_0^\pi d\theta\, \sin\theta\cos^2\theta = \frac{2}{3}, \qquad \int_0^\pi d\theta\, \sin^2\theta\cos\theta = \frac{2}{3}, \qquad \int_0^\pi d\theta\, \sin^4\theta = \frac{3\pi}{8} \tag{12}$$