Quantum Physics III (8.06) Spring 2007
MIDTERM TEST
Thursday March 22, 2007
You have 1 hour and 20 minutes.

There are 5 problems, totalling 80 points. Do all problems.

Answer all problems in the white books provided. Write YOUR NAME on EACH white book you use. Budget your time wisely, using the point values as a guide. Note that shorter problems may not always be easier problems.

No books, notes or calculators allowed.
Some potentially useful information

- Schrödinger equation
  \[ i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle \]
  For an energy eigenstate \( \psi \) of energy \( E \)
  \[ \psi(t) = e^{-\frac{i}{\hbar}Et} \psi(0) \]
  and the Schrödinger equation reduces to an eigenvalue equation
  \[ H \psi = E \psi \]

- Harmonic Oscillator
  \[ \hat{H} = \frac{1}{2m} \hat{p}^2 + \frac{1}{2} m \omega^2 \hat{x}^2 \]
  where
  \[ [\hat{x}, \hat{p}] = i\hbar . \]
  This Hamiltonian can be rewritten as
  \[ \hat{H} = \hbar \omega \left( \hat{N} + \frac{1}{2} \right) \]
  where \( \hat{N} = \hat{a}^\dagger \hat{a} \), and the operators \( \hat{a} \) and \( \hat{a}^\dagger \) are given by
  \[ \hat{a} = \frac{1}{\sqrt{2m\omega\hbar}} (m\omega \hat{x} + i\hat{p}) , \quad \hat{a}^\dagger = \frac{1}{\sqrt{2m\omega\hbar}} (m\omega \hat{x} - i\hat{p}) , \]
  and satisfy
  \[ [\hat{a}, \hat{a}^\dagger] = 1 . \]
  Conversely
  \[ \hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^\dagger) , \quad \hat{p} = \frac{1}{i} \sqrt{\frac{\hbar m\omega}{2}} (\hat{a} - \hat{a}^\dagger) \]
  The action of \( \hat{a} \) and \( \hat{a}^\dagger \) on eigenstates of \( \hat{N} \) is given by
  \[ \hat{a}^\dagger |n\rangle = \sqrt{n + 1} |n + 1\rangle , \quad \hat{a} |n\rangle = \sqrt{n} |n - 1\rangle . \]
  The ground state wave function is
  \[ \langle x|0\rangle = \left( \frac{m\omega}{\pi \hbar} \right)^{1/4} \exp \left( -\frac{m\omega}{2\hbar} x^2 \right) . \]
• Useful integrals
\[
\int_{-\infty}^{+\infty} dx \exp \left(-ax^2\right) = \sqrt{\frac{\pi}{a}}
\]
\[
\frac{1}{2\pi} \int_0^{2\pi} d\phi e^{in\phi} \cos m\phi = \frac{1}{2} (\delta_{m,n} + \delta_{m,-n})
\]

• Particle in an Electric and/or Magnetic Field:
The Hamiltonian for a particle with charge \(q\) in a magnetic field and electric field \(\vec{B} = \vec{\nabla} \times \vec{A}, \vec{E} = -\vec{\nabla} \phi - \frac{1}{c} \partial \vec{A}/\partial t\) is:
\[
H = \frac{1}{2m} \left( \vec{p} - q \frac{\vec{A}}{c} \right)^2 + q\phi
\]
(1)

Note:
\[B_z = \partial_x A_y - \partial_y A_x\]

• Gauge invariance:
If \(\psi(\vec{x}, t)\) solves the Schrödinger equation defined by the Hamiltonian (1), then
\[
\psi'(\vec{x}, t) = \exp \left(-i\frac{q}{\hbar c} f(\vec{x}, t)\right) \psi(\vec{x}, t)
\]
solves the Schrödinger equation obtained upon replacing \(\vec{A}\) by \(\vec{A}' = \vec{A} - \vec{\nabla} f\) and replacing \(\phi\) by \(\phi' = \phi + (1/c) \partial f / \partial t\).

• Time independent perturbation theory:
Suppose that \(H = H_0 + H'\) where we already know the eigenvalues \(E_n^0\) and eigenstates \(|\psi_n^0\rangle\) of \(H^0\):
\[
H_0 |\psi_n^0\rangle = E_n^0 |\psi_n^0\rangle.
\]
Then, the eigenvalues and eigenstates of the full Hamiltonian \(H\) are:
\[
E_n = E_n^0 + H'_{nn} + \sum_{m \neq n} \frac{|H_{nm}'|^2}{E_n^0 - E_m^0} + \ldots
\]
(2)
\[
|\psi_n\rangle = |\psi_n^0\rangle + \sum_{m \neq n} \frac{H_{nm}'}{E_n^0 - E_m^0} |\psi_m^0\rangle + \ldots
\]
(3)
where \(H'_{nm} \equiv \langle \psi_n^0 | H' | \psi_m^0 \rangle\).
If \(H_0\) has degeneracy at \(E_n^0\), first diagonalize \(H'\) in the corresponding degenerate subspace, then use equations (2) and (3). In particular \(|\psi_n^0\rangle\) ("good states") should be one of the eigenvectors of \(H'\) in the degenerate subspace.
1. True or false (10 points)

(a) (2 points) If you dump some matter onto a stable white dwarf star, increasing its mass, its radius becomes smaller.

(b) (2 points) In an Aharonov-Bohm experiment, a solenoid with total magnetic flux $\frac{135\hbar c}{2e}$ will cause the interference fringes to shift by $\pi$.

(c) (2 points) Once you include relativistic and spin-orbit corrections, there are no remaining degeneracies in the hydrogen spectrum.

(d) (2 points) In the weak-field Zeeman effect we treat magnetic field as a small perturbation to the spin-orbit coupling.

(e) (2 points) Momentum operator in $x$-direction $\hat{p}_x$ is a gauge invariant operator.
2. Short answer questions (15 points)

(a) (3 points) Describe how band structure affects the electric conducting property of a solid. What is the difference between an insulator and a semi-conductor?

(b) (2 points) (No derivation required) Consider a free electron moving in a 2-dimensional material of finite area $A$, with a magnetic field $B$ along $z$ direction. What is the degeneracy of a Landau level?

(c) (5 points) Sketch an experiment to measure the Aharonov-Bohm effect. What gauge invariant observable does the Aharonov-Bohm effect measure?

(d) (2 points) Can one use a free electron gas to explain the Integer Quantum Hall Effect? Why?

(e) (3 points) Explain the physical significance of the Chandrasekhar mass for a white dwarf. What happens if one dumps matter onto a white dwarf with a mass smaller than the Chandrasekhar mass to make its mass bigger than the Chandrasekhar mass?
3. Free fermions in a box and spin polarization (18 points)

Consider $N$ non-relativistic spin-$\frac{1}{2}$ fermions with mass $m$ in a cubic box of dimension $L \times L \times L$. Assume that the wave functions satisfy periodic boundary conditions. Imagine that $N$ is very large (say $10^{23}$) so that you can ignore any boundary effects.

(a) (6 points) Describe in words how you would characterize the ground state of the system. Use a plot if you like. Derive the location of the Fermi surface $k_F$.

(b) (5 points) What is the energy density of the ground state?

Now apply a constant magnetic field along $z$ direction to the system. The only effect (which we consider) of the magnetic field is to shift the energy of a fermion by $-\mu B$ ($+\mu B$) if the fermion is spin up (spin down), where $\mu$ is the spin magnetic moment of the fermion.

(c) (3 points) Argue that if $B$ is big enough, in the ground state all fermions are spin up.

(d) (4 points) Find the minimal value of $B$ necessary so that in the ground state all fermions are spin-up.
4. **A particle on a Ring (22 points)**

Consider a particle constrained to move in the $xy$-plane on a circular ring. The only variable of the system is the azimuthal angle $\phi$. The state of the system is described by a wave function $\psi(\phi)$ which must have the property that

$$\psi(\phi + 2\pi) = \psi(\phi)$$

and which should be normalized:

$$\int_0^{2\pi} |\psi(\phi)|^2 d\phi = 1 .$$

The kinetic energy of the particle can be written:

$$H_0 = \frac{L_z^2}{2K}$$

(4)

where $L_z = -i\hbar d/\phi$ and $K$ is a constant. The particle also experiences a potential which can be treated as a small perturbation

$$H' = -\lambda \cos 2\phi .$$

(5)

(a) (4 points) Calculate the eigenvalues and normalized eigenfunctions of $H_0$. Which of the energy levels are degenerate? [Note: Using plane wave basis will simplify your computation below.]

(b) (7 points) Now include the perturbation in this and all parts below. Calculate the lowest non-vanishing correction to the ground state energy due to $H'$.

(c) (3 points) Calculate the ground state wave function to order $\lambda$.

(d) (8 points) Calculate the energy shift(s) due to $H'$ for the next lowest energy state(s) of $H_0$ to first order in $\lambda$. If there are degeneracies, find the correct zero-th order wave functions.

Note: a useful integral for this problem

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi e^{in\phi} \cos m\phi = \frac{1}{2}(\delta_{m,n} + \delta_{m,-n})$$
5. **Particle in a magnetic field (15 points)**

Consider a particle with charge \( q \) and mass \( m \) in a constant magnetic field field \( \vec{B} = (0, 0, B) \). The particle is restricted to move in the \((x, y)\) plane.

Define the velocity operator

\[
\vec{v} = \frac{1}{m} \left( \vec{p} - \frac{q}{c} \vec{A} \right)
\]  

(6)

(a) (3 points) Evaluate \([v_x, v_y]\) (without choosing a gauge).

(b) (6 points) Using the result of (a) to show that the Hamiltonian of the system

\[
H = \frac{1}{2m} \left( \vec{p} - \frac{q}{c} \vec{A} \right)^2
\]  

(7)
can be recast as a simple harmonic oscillator (without choosing a gauge). Determine the energy eigenvalues. **[Note: you will not get much credit for simply writing down the energy eigenvalues. The purpose of this problem is for you to derive the eigenvalues from a gauge invariant starting point.]**

(c) (6 points) Let the wave function of the particle initially (at \( t = 0 \)) have the form

\[
\Psi(x, y; 0) = \psi(x, y)
\]  

(8)

Show that the wave function \( \Psi(x, y; t) \) at time \( t \), aside from an arbitrary phase factor, is periodic in \( t \) with a period \( T \), where \( T = \frac{2\pi}{\omega_L} \) is the period of the classical motion of the particle in the magnetic field and \( \omega_L \) is the classical cyclotron frequency. **[Note: This part can be done independent of (a) and (b).]**