Quantum Physics III (8.06) Spring 2007
Midterm Solution

Mar 23, 2007

1. **True or false (10 points)**

   (a) (2 points) T

   (b) (2 points) T

   (c) (2 points) F

   (d) (2 points) T or F (both correct since the sentence was somewhat ambiguously stated.)

   (e) (2 points) F

2. **Short answer questions (15 points)**

(a) (3 points) If the ground state of a solid have a partially-filled band, it is a conductor. If all bands are either full or empty, the solid is an insulator or semi-conductor. The difference between an insulator and a semi-conductor is that a semi-conductor has a relatively small band gap so that some electrons can be relatively easily excited to a higher band and become conducting electrons.

(b) (2 points) The degeneracy D of a Landau level is $D = \frac{BA}{\Phi_0}$ with $\Phi_0 = \frac{hc}{e}$.

(c) (5 points) One could use a double slit experiment with a solenoid hidden behind the screen to measure the Aharonov-Bohm effect (3 points). The Aharonov-Bohm effect measures the fractional part of (1 point) $\frac{\Phi}{\Phi_0}$ with Φ the total flux of the solenoid (1 point). Note that $\Phi = \oint_C \vec{A} \cdot d\vec{l}$, where C is any curve enclosing the solenoid, and $\Phi_0 = \frac{hc}{q}$ (q is the charge of the particle used in the experiment).

(d) (2 points) No (1 point). In Integer Quantum Hall Effect, the presence of impurities (1 point) are crucial for the appearance of plateaus in the plot of σ_H as a function of the filling ratio, which is not captured by a free electron gas in a electric and magnetic field.

(e) (3 points) The Chandrasekhar mass is the critical mass beyond which the degenerate electron gas pressure can no longer balance the gravity of the star (2 points). If one dumps matter into a white dwarf with a mass smaller than the Chandrasekhar mass to make its mass bigger than the Chandrasekhar mass, it will either collapse to form a neutron star or a black hole (1 point).

3. **Free fermions in a box and spin polartization (18 points)**

Consider N non-relativistic spin-$\frac{1}{2}$ fermions with mass m in a cubic box of dimension $L \times L \times L$. Assume that the wave functions satisfy periodic boundary conditions.

The single particle wave functions are given by

$$\psi_{\vec{n}}(\vec{x}) = \frac{1}{\sqrt{L^3}} e^{i\vec{k}\cdot\vec{x}}, \qquad \vec{k} = \frac{2\pi}{L}(n_1, n_2, n_3), \qquad n_i \in \mathbf{Z} \ . \tag{1}$$

Note that n_i are arbitrary integers. The energy is

$$E_{n_1 n_2 n_3} = \frac{\hbar^2 \vec{k}^2}{2m}$$

which is the same for a spin-up or a spin-down particle.

(a) (6 points)
The ground state of the system is obtained by filling N fermions to lowest possible single-particle states (1). When N is very large, the ground state can be describe by a **sphere** (since n_i in (1) can be both positive or negative) in k-space, inside which the states are filled and outside which the states are unfilled (2 points).
The radius of the sphere k_F is given by

$$2\frac{\frac{4\pi}{3}k_F^3}{(\frac{2\pi}{L})^3} = N \tag{2}$$

where the factor 2 is due to spin degeneracy and $(\frac{2\pi}{L})^3$ is the volume occupied by a state in k-space. Equation (2) leads to

$$k_F = \left(3\pi^2 \frac{N}{V}\right)^{\frac{1}{3}}, \quad V = L^3 \ . \tag{3}$$

(b) (5 points) The ground state energy density is given by

$$\frac{E_0}{L^3} = \frac{1}{L^3} 2 \int_0^{k_F} \frac{4\pi k^2 dk}{(\frac{2\pi}{L})^3} \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{10\pi^2 m} k_F^5 = \frac{\hbar^2}{10\pi^2 m} \left(3\pi^2 \frac{N}{V}\right)^{\frac{5}{3}} \tag{4}$$

where the factor 2 is again due to spin degeneracy.

(c) (3 points) With a constant magnetic field along z direction, the energy for a spin-up particle is

$$E_+ = -\mu B + \frac{\hbar^2 k^2}{2m} \ , \tag{5}$$

while for a spin down particle is

$$E_- = +\mu B + \frac{\hbar^2 k^2}{2m} \ . \tag{6}$$

Thus in finding the ground state we first fill the energy levels of a spin-up particle and start filling the energy levels of a spin-down particle only when k in (5) reaches a value k_B defined by

$$\frac{\hbar^2 k_B^2}{2m} = 2\mu B \tag{7}$$

at which $E_+(k_B) = E_-(k = 0)$.

If B is big enough we may fill all N particles in spin-up states without reaching k_B.

(d) (4 points) The minimal value of B that this happens is determined from

$$2\mu B_{min} = \frac{\hbar^2 \tilde{k}_F^2}{2m} \ , \tag{8}$$

where $\tilde{k}_F$ is the location of Fermi surface for N spin up particles, given by

$$\frac{\frac{4\pi}{3}\tilde{k}_F^3}{(\frac{2\pi}{L})^3} = N \quad \Rightarrow \quad \tilde{k}_F = \left(6\pi^2 \frac{N}{V}\right)^{\frac{1}{3}} \ . \tag{9}$$

Note in the LHS of the first equation in (9) there is no factor 2 compared with (2), since here we are only filling spin-up particles.

Thus we find

$$B_{min} = \frac{\hbar^2}{4m\mu}\left(6\pi^2 \frac{N}{V}\right)^{\frac{2}{3}} \tag{10}$$

4. **Particle on a Ring (22 points)**

(a) (4 points) With periodic boundary condition, the energy-eigenstates are

$$\psi_n^{(0)} = \frac{1}{\sqrt{2\pi}} e^{in\phi}, \qquad n = 0, \pm 1, \pm 2, \cdots \tag{11}$$

with energy

$$E_n^{(0)} = \frac{\hbar^2 n^2}{2K} \tag{12}$$

States with $n \neq 0$ are doubly degenerate.

(b) (7 points)
The 1st order correction to ground state energy is (2 points)

$$E_0^{(1)} = \langle \psi_0 | H' | \psi_0 \rangle = -\frac{\lambda}{2\pi} \int_0^{2\pi} d\phi \, \cos 2\phi = 0 \ . \tag{13}$$

The second order corrections is given by

$$E_0^{(2)} = \sum_{n=\pm 1}^{\infty} \frac{|\langle \psi_0^{(0)} | H' | \psi_n^{(0)} \rangle|^2}{E_0^{(0)} - E_n^{(0)}} \tag{14}$$

Note that (2 points)

$$\langle \psi_0^{(0)} | \cos 2\phi | \psi_n^{(0)} \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\phi \, e^{in\phi} \cos 2\phi = \frac{1}{2} (\delta_{n,2} + \delta_{n,-2}) \tag{15}$$

Using (15) in (14) we find that (3 points)

$$E_0^{(2)} = 2 \cdot \frac{\lambda^2}{4} \frac{1}{0 - \frac{4\hbar^2}{2K}} = -\frac{\lambda^2 K}{4\hbar^2} \tag{16}$$

(c) (3 points) The first order correction to ground state wave function is

$$\psi_0^{(1)} = \sum_{n=\pm 1}^{\infty} \frac{\langle \psi_n^{(0)} | H' | \psi_0^{(0)} \rangle}{E_0^{(0)} - E_n^{(0)}} \psi_n^{(0)} = \frac{\lambda K}{4\hbar^2} \left(\psi_2^{(0)} + \psi_{-2}^{(0)} \right) \tag{17}$$

where we have used (15).

(d) (8 points) The next lowest energy states of H_0 are $\psi_{\pm 1}^{(0)}$ which are doubly degenerate. We thus need to use degenerate perturbation.
Note that (4 points)

$$\langle \psi_1^{(0)} | \cos 2\phi | \psi_1^{(0)} \rangle = \langle \psi_{-1}^{(0)} | \cos 2\phi | \psi_{-1}^{(0)} \rangle = 0 \tag{18}$$

while

$$\langle \psi_1^{(0)} | \cos 2\phi | \psi_{-1}^{(0)} \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\phi \, e^{-2i\phi} \cos 2\phi = \frac{1}{2} \tag{19}$$

Thus we find that in this degenerate space

$$H' = \frac{1}{2}\begin{pmatrix} 0 & -\lambda \\ -\lambda & 0 \end{pmatrix} \tag{20}$$

Thus the energy shift are $E_+^{(1)} = -\frac{1}{2}\lambda$, corresponding to (2 points)

$$\psi_+^{(0)} = \frac{1}{\sqrt{2}}(\psi_1^{(0)} + \psi_{-1}^{(0)}) = \frac{1}{\sqrt{\pi}}\cos\phi \tag{21}$$

and $E_-^{(1)} = +\frac{1}{2}\lambda$, corresponding to (2 points)

$$\psi_-^{(0)} = \frac{1}{\sqrt{2}}(\psi_1^{(0)} - \psi_{-1}^{(0)}) = \frac{i}{\sqrt{\pi}}\sin\phi \tag{22}$$

5. **Particle in a magnetic field (15 points)**

Consider a particle with charge q and mass m in a constant magnetic field field $\vec{B} = (0, 0, B)$. The particle is restricted to move in the (x, y) plane.

Define the velocity operator

$$\vec{v} = \frac{1}{m}\left(\vec{p} - \frac{q}{c}\vec{A}\right) \tag{23}$$

(a) (3 points)

$$[v_x, v_y] = \frac{1}{m^2}[p_x - \frac{q}{c}A_x, p_y - \frac{q}{c}A_y] = \frac{i\hbar q}{m^2 c}(\partial_x A_y - \partial_y A_x) = \frac{i\hbar q B}{m^2 c} = i\frac{\hbar\omega_L}{m} \tag{24}$$

where we used

$$B = B_z = \partial_x A_y - \partial_y A_x \tag{25}$$

and the definition

$$\omega_L = \frac{qB}{mc} \tag{26}$$

(b) (6 points) The Hamiltonian of the system

$$H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 = \frac{1}{2}m(v_x^2 + v_y^2) \tag{27}$$

Equation (24) implies that v_x and v_y behave like conjugate position and momentum. In particular introduce

$$X = \frac{v_x}{\omega_L}, \qquad P = m v_y \tag{28}$$

then

$$[X, P] = i\hbar \tag{29}$$

Equation (27) can be written in terms of X and P as

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega_L^2 X^2 \tag{30}$$

which is the Hamiltonian for a Harmonic oscillator with frequency ω_L. Thus we find energy eigenvalues

$$E_n = (n + \frac{1}{2})\hbar\omega_L, \qquad n = 0, 1, 2\cdots \tag{31}$$

(c) (6 points) Expand $\psi(x, y)$ in terms of a complete sets of energy eigenstates

$$\psi(x, y) = \sum_\alpha c_\alpha \psi_\alpha$$

where $\sum_\alpha$ sums over the complete set of energy eigenstates. Then at time t we find that

$$\Psi(x, y, t) = \sum_\alpha c_\alpha e^{-\frac{i}{\hbar}E_n t}\psi_\alpha$$

with E_n given by (31). Now let $t = lT$ with l an integer and $T = \frac{2\pi}{\omega_L}$, we then find that

$$\Psi(x, y; lT) = (-1)^l \psi(x, y)$$

Thus up to an irrelevant overall phase factor $(-1)^l$ (which is universal for all wave functions) $\Psi(x, y; t)$ is periodic in t with a period T.