# MIT Quantum Theory Notes

## Supplementary Notes for MIT's Quantum Theory Sequence



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# Natural Units
# and the Scales of Fundamental Physics

# Contents

# 1 Introduction

Clearly different units are "natural" for different problems. Car mechanics like to measure power in *horsepower*, electrical engineers prefer *watts* and particle physicists prefer $\text{MeV}^2$. A good choice of units can make the magnitudes of interesting quantities more palatable. An planetary scientist measures distances in astronomical units ($au$), 1 $au$=149597870660(20) meters[1], the mean Earth-Sun distance. It would be cumbersome for her to use microns, and of course, the reverse is true for a condensed matter physicist.

Each to her own... It seems like a pretty dull subject. However, in the realm of modern physics a careful examination of the choice of units leads to some useful (even profound) insights into the way the Universe works. In this chapter I briefly review the *MKS* and *cgs* systems. Then I introduce the system that quantum physicists call *Natural Units*. Although it sounds arrogant, these really are the *natural units* for the micro-world. To convince you of this, I go on to describe some consequences of the use of natural units to describe relativistic and quantum phenomena.

# 2 The *MKS* and *cgs* Systems of Units (very briefly)

In the *MKS* system of units mechanical quantities are expressed using the kilogram (kg), meter (m), and second (sec), as the fundamental units of mass ($m$), length ($\ell$) and time ($t$). Velocities are quoted as so many "meters per second", forces as so many "kg m/sec$^2$". If the language becomes cumbersome, new names are introduced: so, for example, the "Newton" is defined to be 1 kg m/sec$^2$, but introducing new names does not change the {*kilogram meter second*} at the core.

When dealing with electromagnetic phenomena, the *MKS* system introduces a new fundamental unit of charge, the *Coulomb*, which we can

[1] The notation 149597870660(20) is to be read 149597870660±20, and is used when quoting a number to the present limit of experimental accuracy. Otherwise all numbers and calculations in this section are quoted to four significant figures. The precision measurement of fundamental constants is an epic saga in modern physics — every digit in these numbers represents years, if not lifetimes, of imaginative and difficult research. You can find precision measurements of fundamental constants and conversion factors in the tables provided by the Particle Data Group, available on-line at `http://pdg.lbl.gov/`

think of as the charge on approximately $6.24\times10^{18}$ electrons. In *MKS* units Coulomb's Law,

$$F = \frac{1}{4\pi\epsilon_0}\frac{Q_1Q_2}{r^2} \quad (MKS) \tag{1}$$

must include a constant of proportionality (called, for historical reasons, $1/4\pi\epsilon_0$), which measures the force (in Newtons) between two one-Coulomb charges separated by one meter. Notice that the constant of proportionality has appeared because we insisted on introducing our own favorite unit of charge, the Coulomb. Had we chosen a different unit of charge, the proportionality constant would have been different. Wouldn't it be great if we could choose a unit of charge so that the constant of proportionality was unity? That is precisely what is done in the *cgs* system, which makes the slight additional change of measuring mass in grams and distance in centimeters.

In the *cgs* system *all* physical quantities — not just mechanical — are expressed in terms of grams (g), centimeters (cm) and seconds (sec). It is easy to see how the quantities that arise in meachanics, like momentum, energy, or viscosity have units that are derived from defining equations. Because $p = mv$, $E = \frac{1}{2}mv^2 + \ldots$, and $dF_x/dA = \eta\partial v_x/\partial y$ we know that[2]

$$\begin{aligned}
[\text{momentum}] &= m\ell t^{-1} && \text{gm cm sec}^{-1} \\
[\text{energy}] &= m\ell^2 t^{-2} && \text{gm cm}^2\text{ sec}^{-2} \\
[\text{force}] &= m\ell t^{-2} && \text{gm cm sec}^{-2} \\
[\text{viscosity}] &= m\ell^{-1} t^{-1} && \text{gm cm}^{-1}\text{ sec}^{-1}\ .
\end{aligned} \tag{2}$$

Of course practitioners introduce convenient abbreviations: For example,

- The *cgs* unit of force is a *dyne*, equal to one gm cm $\text{sec}^{-2}$;
- The *cgs* unit of energy is a *erg*, equal to one gm $\text{cm}^2$ $\text{sec}^{-2}$;
- The *cgs* unit of viscosity is a *poise*, equal to one gm $\text{cm}^{-1}$ $\text{sec}^{-1}$.

But the dyne, erg, or poise, have no fundamental significance: everything is just grams, centimeters, and seconds. From a *cgs* point of view, any other unit used in mechanics, like a *foot*, an *atmosphere* or an *acre* merely represents a convenient short hand for so-many $\text{gm}^a\text{cm}^b\text{sec}^c$, where the exponents, $a$,

[2] In the subsequent equations $[x]$ is to be read "the dimensions of $x$."

$b$, and $c$ are chosen to give the correct dimensions. It is clear that all the quantities encountered in mechanics can be expressed in terms mass, length, and time.

The real power of the *cgs* approach becomes apparent when we leave the realm of mechanics. Consider, for example, electrodynamics. When a new concept such as *electric charge* is first encountered, it seems necessary to introduce a new unit to measure its quantity. In the case of charge, both the *Coulomb* and the *Faraday* were introduced in this way before the laws of electromagnetism were known.

However, the need for an independent unit for electric charge went away when the dynamical laws of electrostatics were worked out. Coulomb's Law enables us to measure charge *using the same units we used in mechanics*, $\ell$, $m$, and $t$. Coulomb's Law tells us that the force produced by charges at a fixed separation is proportional to the product of the charges and inversely proportional to their separation squared,

$$F_{\text{coulomb}} \propto \frac{q_1 q_2}{r^2} \tag{3}$$

This affords us the opportunity to define a unit of charge within the existing *cgs* system. Simply define one *cgs* unit of charge as *the charge necessary to produce a force of one dyne at a separation of one centimeter* from an equal charge. Then in this system, the proportionality in eq. (3) becomes equality,

$$F_{\text{coulomb}} = \frac{q_1 q_2}{r^2} \quad (cgs) \tag{4}$$

The *cgs* unit of charge, known as the *statcoulomb* or *esu*[3], is convenient because it eliminates the need for the constant of proportionality, $1/4\pi\epsilon_0$, that appeared in eq. (1). Even better, it tells us that charge can be measured in the same units of mass, length, and time, that were sufficient for mechanics. To see this consider the balance of dimensions in eq. (4),

$$\begin{aligned}
[\text{force}] &= [\text{charge}]^2/[r]^2, \text{ so} \\
[\text{charge}] &= [\text{force}]^{1/2} \otimes \ell \\
&= [m\ell t^{-2}]^{1/2} \otimes \ell \\
&= m^{1/2}\ell^{3/2}t^{-1}
\end{aligned} \tag{5}$$

[3] *esu* $\equiv$ *electrostatic unit*

So charge has dimensions $m^{1/2}\ell^{3/2}t^{-1}$ in the *cgs* system and is measured in $\text{gm}^{1/2}\text{cm}^{3/2}\text{sec}^{-1}$(!) Since it would be cumbersome to refer to the units of charge as the "$\text{gm}^{1/2}\text{cm}^{3/2}\text{sec}^{-1}$", this unit is given its own name, the *esu*. Of course it can also be expressed as the charge of so many electrons ( $2.08194345(18) \times 10^9$ at the present limit of precision) however the *esu* has a fundamental connection to the *cgs* system that the other units of charge do not.

The "trick" here was to write Coulomb's Law *without* any constant of proportionality. This is the *cgs* algorithm for introducing new concepts into physics without introducing new units: Simply write down the law relating the new concept to already known quantities *without a constant of proportionality*. Then let the law define the units. This works as long as the new quantity under study appears in a mathematical equation that relates it to known quantities. At this moment in physics history, all the quantities we measure can be expressed in terms of mass, length, and time. Problem 6 explores how this comes about when a new phenomenon is discovered.

The *MKS* system is different. A new, *ad hoc* unit, the Coulomb, is introduced, and a proportionality constant, $1/4\pi\epsilon_0$, is introduced into Coulomb's Law to preserve the meaning of independently defined units. For this reason the *MKS* system is not used much in fundamental physics, although it is most convenient for engineering applications where units matched to practical applications are highly desirable.

To make sure we understand the *cgs* approach, and to introduce a small elaboration, let's study some further examples from electromagnetism. The definition of electric field tells us its units: $\vec{F} = e\vec{E}$. Given the units of force and charge that we have worked out, we find that $\vec{E}$ has dimensions $m^{1/2}\ell^{-1/2}t^{-1}$ and its units are $\text{gm}^{1/2}\ \text{cm}^{-1/2}\ \text{sec}^{-1}$. A slight complication arises when magnetism is introduced. The *cgs* units for the magnetic field can be determined from the Lorentz Force Law,

$$\vec{F} = e\vec{E} + e\vec{v} \times \vec{B}. \tag{6}$$

We could, of course, change the units for the magnetic field, $B$, by putting a (dimensionful) constant of proportionality in front of the second term on the right hand side of eq. (6), although this seems to run counter to the spirit of the *cgs* approach. When electromagnetic radiation is important, however, it is very convenient to use a system where the electric and magnetic fields are measured *in the same units*. This can be accomplished if a constant of proportionality with dimensions *1/velocity* is introduced into eq. (6). Electrodynamics offers a natural candidate for this velocity: $c$ — the speed of light. So a very useful extension of the *cgs* system to magnetic effects uses

$$\vec{F} = e\vec{E} + e\frac{\vec{v}}{c} \times \vec{B} \tag{7}$$

as the Lorentz Force Law. Then $\vec{E}$ and $\vec{B}$ have the same units. This particular way of extending the *cgs* system to electrodynamics is known as the *Gaussian* system of units. It requires introducing a few other factors of $c$ into common electromagnetic formulas. Here is a sampling of equations of electrodynamics written in *cgs* units[4]

$$\begin{aligned}
\vec{F}_{12} &= \frac{e_1 e_2}{r_{12}^2}\hat{r}_{12} && \text{Coulomb's Law}\\
\vec{\nabla}\cdot\vec{E} &= 4\pi\rho && \text{Gauss's Law}\\
\vec{F} &= e\vec{E} + \frac{e}{c}\vec{v}\times\vec{B} && \text{Lorentz's Force Law}\\
V &= \int d\vec{l}\cdot\vec{E} && \text{Definition of voltage}\\
V &= IR && \text{Ohm's Law}\\
R &= \rho\times\text{length/area} && \text{Resistance from resistivity}\\
V &= -\frac{L}{c}\frac{dI}{dt} && \text{Faraday's Law}\\
\vec{\nabla}\times\vec{B} &= \frac{4\pi}{c}\vec{j} && \text{Ampere's Law}
\end{aligned} \tag{8}$$

It doesn't matter if you are not familiar with all these equations because I won't be using them in great detail. From them you can deduce the dimensions of commonly encountered quantities in the *cgs* system,

$$\begin{aligned}
[\text{resistivity}] &= t\\
[\text{resistance}] &= \ell^{-1}t\\
[\text{inductance}] &= \ell^{-1}t^2\\
[\text{magnetic field}] &= m^{1/2}\ell^{-1/2}t^{-1}
\end{aligned} \tag{9}$$

[4] For a more comprehensive discussion of electromagnetic units see the Appendix on units in J. D. Jackson, *Classical Electrodynamics, 3rd Edition*, where, for example, the factor of $c$ in Faraday's Law is discussed.

Like the electric charge, the *cgs* units of electric and magnetic fields are cumbersome. They are usually replaced by the *gauss*, defined by 1 gauss = 1 $\text{gm}^{1/2}\ \text{cm}^{-1/2}\ \text{sec}^{-1}$.

There is more to be said about the *cgs* system. For example, there is a common variant in which the factor of $4\pi$ that decorates the differential form of Gauss's law in eq. (8) is removed by redefining the unit of charge. As you can imagine, the situation can get complicated. Fortunately we are not heading in this direction. If you are interested, you can find a (thankfully) short but illuminating discussion in the Appendix to Jackson's book already noted.

# 3 Natural Units

The *MKS* and *cgs* systems are convenient, practical systems for most macroscopic applications. When we leave the scale of human dimensions to study very small sizes and very energetic processes, they are no longer so natural. Centimeters, grams and seconds are not particularly appropriate units for problems where relativity and quantum mechanics are important. This is reflected in the appearance of large exponents in quantities like the speed of light ($2.99792458 \times 10^{10}$ cm $\text{sec}^{-1}$) and Planck's constant ($6.6260693(11)^{-27}$ gm $\text{cm}^2$ $\text{sec}^{-1}$) expressed in *cgs* units. In the micro-world, the fundamental constants $c$ and $\hbar$ set natural scales for velocity and action.[5] *Natural Units* where velocity and action are measured in terms of $c$ and $\hbar$ respectively, have won wide acceptance among atomic, nuclear, particle and astrophysicists, and theorists of all kinds. They make dimensional analysis very simple and even suggest the natural time, distance, and energy scales of fundamental interactions. The use of natural units is surrounded with some unnecessary confusion and mystery because of the physicists' habit of abbreviating their notation and expressing *all physical quantities in terms of electron-volts (to the appropriate power).*

In the *cgs* system mass, length, and time sufficed to give us units for all physical quantities. Physical laws can be used to relate the units of anything else to these three. It seems self-evident that the units of *mass, length and*

[5] Remember that Planck's constant has units of *action*, which are the same as position×momentum. As a reminder, consider the Bohr-Sommerfeld quantization condition, $\oint pdq = n\hbar$.

*time* are fundamental and independent.[6] We do not know any fundamental physical law relating mass, length, and time, that would enable us to use the units of one to measure the other. However, there is no reason to choose mass, length, and time to be the three building blocks for our units. We could instead choose any three other quantities, provided any combination of mass, length, and time can be re-expressed in terms of them. Natural units choose instead *velocity*, *action*, and *energy* as the building blocks. Any quantity that can be expressed in terms of fundamental units of mass, length, and time, can just as well be expressed in terms of some choice of fundamental units of velocity, action, and energy.

There are actually wonderful reasons to choose velocity and action to be two of our fundamental units: $c$, the speed of light, is a natural unit of velocity and $\hbar$, Planck's constant (divided by $2\pi$), is a natural unit of action. So if we use these units, we will typically have very nice expressions for things that use relativity and quantum mechanics in an essential way. Let us, for a brief time and for pedagogical purposes, call the speed $c$ a new unit of velocity, the *einstein* (abbreviated $c$), and call the action $\hbar$ a new unit of action, the *planck* (abbreviated $\hbar$). To complete a new set of basic units we need only decide on another quantity, as long as it is linearly independent from velocity and action. The conventional choice is energy, and the usual choice of unit is the *electron-volt* (abbreviated eV). Everywhere where grams, centimeters, and seconds appear in the *cgs* system we use the abbreviations gm, cm, and sec. Likewise, everywhere where the *einstein* — our fundamental unit of velociy — appears in the natural unit system, we use the abbreviation $c$. Where the *planck* appears, we use the abbreviation $\hbar$, and where the *electron-volt* appears, we use the abbreviation eV.

If we express all physical quantities in terms of $c$, $\hbar$, and eV, then we are using *natural units.* At the core that is all there is to it. Any quantity expressed in *cgs* units, *ie.* as $\text{gm}^a\ \text{cm}^b\ \text{sec}^c$, can be converted to natural units

[6] It is always dangerous to say something is "self-evident", and this is no exception. See Problem 7 for the reason why.

using the basic conversion factors between natural and *cgs* systems,[7]

$$\begin{aligned} 1\ c &= 2.99792458 \times 10^{10}\,\text{cm sec}^{-1} \\ 1\ \hbar &= 1.05457266(63) \times 10^{-27}\,\text{gm cm}^2\text{sec}^{-1} \\ 1\ \text{eV} &= 1.60217733(49) \times 10^{-12}\,\text{gm cm}^2\text{sec}^{-2} \quad . \end{aligned} \tag{10}$$

and their (less familiar) inverses,

$$\begin{aligned} 1\ \text{sec} &= 1.51926689(xx) \times 10^{15}\ \hbar\ \text{eV}^{-1} \\ 1\ \text{cm} &= 5.06772886(xx) \times 10^{4}\ \hbar\ c\ \text{eV}^{-1} \\ 1\ \text{gm} &= 5.60958616(xx) \times 10^{32}\ \text{eV}\ c^{-2} \end{aligned} \tag{11}$$

Some other useful conversion factors are

$$\begin{aligned} \hbar c &= 197.327053(59)\,\text{MeV fm} \\ \hbar &= 6.5821220(20) \times 10^{-22}\,\text{MeV sec} \\ e^2 &= [137.0359895(61)]^{-1}\ \hbar c \text{ where } e \text{ is the electron charge} \\ 1\ \text{gauss} &= 1\ \text{gm}^{1/2}\ \text{cm}^{-1/2}\ \text{sec}^{-1} \end{aligned} \tag{12}$$

where $1\,\text{MeV} = 10^6\,\text{eV}$, and $1\ \text{fm} = 10^{-13}\,\text{cm}$ is a *femptometer* or a *fermi*, and remember that $1\ \text{erg} = 1\ \text{gm cm}^2\ \text{sec}^{-2}$ and $1\ \text{gauss} = 1\ \text{gm}^{1/2}\ \text{cm}^{-1/2}\ \text{sec}^{-1}$. A quirky statement that causes much confusion is that "$\hbar c$ is equal to unity in natural units." Since $\hbar$ is "one planck" and $c$ is "one einstein", this statement is really no different than saying that "one gram × one centimeter is equal to unity in *cgs* units"!

It is useful to have a generic relation between the units of a quantity in the *cgs* system and in natural units. Consider some quantity $D$ which is known in *cgs* units, $[D] = \text{gm}^a\ \text{cm}^b\ \text{sec}^c$. Apparently the dimensions of $D$ are $[D] = m^a\ell^b t^c$. In natural units, $[D] = [\text{c}]^\alpha[\hbar]^\beta[\text{eV}]^\gamma$. The powers of $c$, $\hbar$, and $eV$ are fixed by using the relations

$$\begin{aligned} [\text{c}] &= \ell t^{-1} \\ [\hbar] &= m\ell^2 t^{-1} \\ [\text{eV}] &= m\ell^2 t^{-2} \end{aligned} \tag{13}$$

[7]Actually, the fact that $c$ is quoted without errors in eq. (10) is a signal that it is used as a fundamental unit in the *cgs* system too. The *second* is defined in terms of the frequency of a specific atomic spectral line, and $2.99792458\times10^{10}$ *centimeters* is how far light goes in a second.

and demanding dimensional consistency between the two systems, $[D] = m^a \ell^b t^c = c^\alpha \hbar^\beta E^\gamma$. The result is,

$$\begin{aligned} \alpha &= b - 2a \\ \beta &= b + c \\ \gamma &= a - b - c \end{aligned} \tag{14}$$

To see how this works let's express the mass of the electron in natural units. We start from $m_e = 9.109 \times 10^{-28}$ gm, and use the third of eqs. (11),

$$\begin{aligned} m_e &= 9.109 \times 10^{-28} \text{ gm} \times (5.610 \times 10^{32} \text{ eV } c^{-2}/ \text{ gm}) \\ &= 0.5110 \times 10^6 \text{ eV } c^{-2} \end{aligned} \tag{15}$$

I hope you will recognize $\sim 0.5$ MeV as the electron's rest energy. In natural units any mass is given by the equivalent rest energy.

Now comes the final, and really the only confusing step: By convention *we drop the factor of* $c^{-2}$ when we write a mass in natural units, and write merely $m_e \simeq 0.5110$ MeV. When a physicist writes a mass in units of MeV, the necessary factors of $c$ required to restore dimensional consistency are "understood". In fact, when any quantity is written in natural units, *all factors of* $\hbar$ *and* $c$ *are suppressed.* So although the natural unit system is based on three fundamental units, the *einstein* — the fundamental unit of velocity, the *planck* — the fundamental unit of action, and the electron-volt — the fundamental unit of energy, the first two are simply dropped in all expressions; which of course explains why you have never heard of them as units. With eqs. (13)–(14) we can proceed to express any physical quantity in natural units, following the same steps we used for mass. Here are some examples:

$$\begin{aligned} [\text{mass}] &= \text{eV } c^{-2} \\ [\text{time}] &= \text{eV}^{-1} \hbar \\ [\text{length}] &= \text{eV}^{-1} \hbar\, c \\ [\text{momentum}] &= \text{eV } c^{-1} \\ [\text{force}] &= \text{eV}^2 \hbar^{-1} c^{-1} \\ [\text{pressure}] &= \text{eV}^4 \hbar^{-3} c^{-3} \\ [\text{charge}^2] &= \hbar c \\ [\text{magnetic field}] &= \text{eV}^2 \hbar^{-3/2} c^{-3/2} \quad . \end{aligned} \tag{16}$$

When the factors of $c$ and $\hbar$ are suppressed, all of these things are measured in electron-volts to some power. So in natural units the units of mass, energy, and momentum are all $eV$, the units of length and time are both $eV^{-1}$, electric charge is dimensionless, and the units of force and magnetic field are both $eV^2$. Many physically different quantities are measured in the same units. That's why natural units are not for children. You have to be sophisticated enough to figure out the meaning from the context. As you become familiar with the system, it becomes easy.

The natural question at this point is "Why bother?". Aren't *cgs* units good enough? The answer is "No." The next subsection is devoted to convincing you of this.

# 4 Advantages of the natural system of units

Natural units have a practical advantage: they are simple and it is easy to convert back to *cgs* or *MKS* when necessary. They also have deeper advantages, which are more important: they are "natural" in the sense that they provide scales that are appropriate to problems in relativistic or quantum physics, and they are perfectly suited to dimensional analysis, which can provide profound insights. These are big claims that need to be demonstrated.

## 4.1 Simplicity

The first great advantage — and the great confusion for non-experts — is that *all physical quantities measured in units of electron volts.* Really, of course, our units are {*einsteins plancks electron-volts*}, but we have suppressed the factors of $c$ and $\hbar$.

We could have done the same thing in the *cgs* system too. We could, for example, suppress the labels *sec* and *cm* and measure all quantities as some power of a fundamental unit of mass, the *gram*. This is not done for two reasons: first, because there is nothing particularly fundamental about one second or one centimeter so we are not eager to suppress the label which tells us that time was measured in seconds and length in centimeters; and second, because we are used to having a *different* set of units for every different physical quantity. For example, momentum and energy have different units in the *cgs* system (*cf.* eq. (2)), but they would both be measured in grams if we suppressed *cm* and *sec*. If you quoted an answer to a calculation in grams, you would have to tell your reader whether it was a momentum or an energy before he would be able to evaluate it in *cgs* units.

In the case of natural units the first disadvantage is eliminated: $\hbar$ and $c$ *are* natural units for action and velocity in fundamental physics. The second disadvantage remains, but it is outweighed by the convenience of measuring all quantities in the same units. Of course, one must be careful to specify the physical quantity of interest to avoid confusing things measured in the same powers of eV. Once the nature of the quantity is known, the problem of converting back from natural units to *cgs* units reduces to use of the conversion factors in eqs. (11) and (12).

To get used to the specification of many different quantities in units of eV consider all the physical quantities defined in eq. (16) associated with $m_e c^2 \simeq 0.5110$ MeV.[8]

- **Mass**

  We already know the mass which is 0.5110 MeV in natural units, 0.5110 MeV$/c^2 = m_e = 9.109 \times 10^{-28}$ gm, the mass of the electron.

- **Momentum**

  $p = E/c$, so in this case

$$\begin{aligned} p &= 0.5110 \text{ MeV} \times (1.602 \times 10^{-12} \text{ erg/eV})/(2.998 \times 10^{10} \text{ cm/sec}) \\ &= 2.730 \times 10^{-17} \text{ gm cm sec}^{-1}\,, \end{aligned} \tag{17}$$

  a very small number in *cgs* units.

- **Length**

  According to eqs. (16), $\ell = \hbar c/E$. In this case, since $E = m_e c^2$, the length is $\ell = \hbar c/m_e c^2 = \hbar/m_e c$, which you might recognize as the Compton wavelength of the electron. Here the first conversion factor, $\hbar c \simeq 197.3$ MeV fm, comes in handy

$$\ell = (197.3 \text{ MeV fm})/(0.5110 \text{ MeV}) = 3.861 \times 10^{-11} \text{ cm} \tag{18}$$

- **Time**

[8] Only four significant figures are retained throughout.

This is easy: once we know the length $(0.5110 \text{ MeV})^{-1} = 3.861 \times 10^{-11}$ cm, the associated time is $\ell/c$ ($= 1.288 \times 10^{-21}$ sec) as required by eq. (16). This is the time it takes light to travel the electron's Compton wavelength.

- **Pressure**

  $P = (0.5110 \text{ MeV})^4/\hbar^3 c^3$ is a pressure. Using the versitile conversion factor, $\hbar c \simeq 197.3$ MeV fm, we find

$$\begin{aligned} P &= (0.5110 \text{ MeV})^4/(197.3 \text{ MeV fm})^3 \\ &= 8.878 \times 10^{-9} \text{ MeV/fm}^3 (1.602 \times 10^{-12} \text{ erg/eV})(10^{13} \text{ fm/cm})^3 \\ &= 1.422 \times 10^{24} \text{ erg/cm}^3 = 1.422 \times 10^{24} \text{ dynes/cm}^2 \\ &= 1.400 \times 10^{18} \text{ atmospheres} \end{aligned} \tag{19}$$

  where I have used the conversion factor 1 atmosphere $= 1.013 \times 10^6$ dynes cm$^{-2}$ to express the answer in everyday units. Notice how large a pressure is associated with the energy scale defined by the electron mass.

- **Electric field**

  According to eq. (16), $|\vec{E}| = (0.5110 \text{ MeV})^2/(\hbar c)^{3/2}$ is an electric field strength. We would like to express it in terms of something familiar, like volts per centimeter. First, note that by definition 1 eV$/e = 1$ volt, where $e$ is the electron charge. According to eq. (12), $(\hbar c)^{1/2} \simeq \sqrt{137.0}\, e \simeq 11.71e$, so we can write,

$$\begin{aligned} |\vec{E}| &= (.5110 \text{ MeV})^2/\left(\hbar c \times (\hbar c)^{1/2}\right) \\ &= (.5110 \text{ MeV})^2/\left(197.3 \text{ MeV fm} \times 11.71\, e\right) \\ &= 1.132 \times 10^{15} \text{ Volts cm}^{-1} \end{aligned} \tag{20}$$

- **Magnetic field**

  Although a magnetic field could also be expressed in volts/cm, it is more conventional to use *gauss*. So, for completeness we use the final conversion factor from eq. (16) to convert

$$\begin{aligned}
|\vec{B}| &= (.5110 \text{ MeV})^2/\,(\hbar c)^{3/2} \\
&= (.5110 \text{ MeV})^2/(197.3 \text{ MeV fm})^{-3/2} \\
&= 9.422\times 10^{-5} \text{ MeV}^{1/2} \text{ fm}^{-3/2}(1.602\times 10^{-12} \text{ erg/eV})^{1/2} \\
&\quad \times(10^{-13} \text{ cm/fm})^{-3/2} \\
&= 3.771\times 10^{12} \text{ gm}^{1/2} \text{ cm}^{-1/2}\text{sec}^{-1} \\
&= 3.771\times 10^{12} \text{ gauss}
\end{aligned} \tag{21}$$

## 4.2 Naturalness

$\hbar$ and $c$ set the scale for quantum mechanics and relativity. When we use them as the basis of our unit system we naturally incorporate fundamental properties of the system under study. Natural units reduce the basic equations of relativity and quantum mechanics to identities — $E = mc^2$, $E = pc$, $\lambdabar_{\text{deBroglie}} = \hbar/p$, $E = \hbar\omega$, $\lambdabar_{\text{Compton}} = \hbar/mc$.

These are more than mere formal relations. Each of them has physical implications in elementary processes. I will put the factors of $\hbar$ and $c$ that are usually suppressed in parentheses:

- When an electron and positron annihilate into two photons. The energy of each photon in the $e^+e^-$ center-of-mass is $m_e(c^2)$ and the frequency is $\omega = m_e(c^2/\hbar)$.

- When an electron of momentum $p$ scatters from a heavy target, the maximum momentum transferred to the target is $2p$. By observing the scattered electrons it is possible to resolve structure within the target down to distances of order $\Delta x \sim (\hbar)/2p$.

- If an external agent were able to confine an electron in a region of space of size of order its Compton wavelength, it would have to exert a pressure of order $m_e^4/((\hbar c)^3)$.

- In relativistic quantum mechanics particles exert forces on one another by exchanging other particles over distances allowed by the uncertainty principle. If two particles exchange a third that has mass $\mu$. Its energy is at least $\mu(c^2)$. The fluctuation can last no longer than $\Delta t \sim (\hbar)/\mu(c^2)$, in which time the particle can travel a distance no more than $\Delta x \sim c\Delta t$. So the range of the force is $\Delta x \sim (\hbar)/\mu(c)$, which is the exchanged particle's Compton wavelength.

Even the seemingly more obscure relations have direct physical implications. As an example consider the phenomenon of electron positron pair creation in a strong electric field, known picturesquely as "ionizing the vacuum." The idea is as follows: In a very strong external electric field a quantum fluctuation can lead to electron-positron pair creation. This limits the possible strength of an electric field. It is exactly analogous to the breakdown of an insulator that causes sparking in a condenser and limits the electric field it can sustain, except here the oppositely charged pairs are created by a quantum fluctuation in the vacuum rather than by ionizing the atoms of an dielectric material. A quantum fluctuation can create an electron positron pair with energy $\Delta E \gtrsim 2m_e c^2$ provided the fluctuation lives less than the time $\Delta t \lesssim \hbar/\Delta E$. In that time, the electron and positron can separate by a distance of order $\Delta x \sim c\Delta t$. As they separate they gain energy $e\mathcal{E}\Delta x$, in the electric field with strength $\mathcal{E}$. If they gain enough energy to compensate for their rest mass, they no longer have to annihilate: like Pinnocchio, they can become real. The condition for real $e^+e^-$ pair creation is therefore that the electric field be greater than a critical value, $\mathcal{E}_{\rm crit}$ determined by

$$e(c\frac{\hbar}{2m_e c^2})\mathcal{E}_{\rm crit} \sim 2m_e c^2 \tag{22}$$

or

$$\mathcal{E}_{\rm crit} \sim 4\frac{m_e^2 c^4}{\hbar c e} = 4(11.71)\frac{m_e^2 c^4}{(\hbar c)^{3/2}} \tag{23}$$

where I have used $e^2/\hbar c \simeq 1/137$. So the critical field is a simple multiple of the electric field specified by the electron rest energy squared.[9]

The final reason that I advertised for adopting natural units was the emergence of important insights from dimensional analysis that is natural in this unit system. The next section gives a series of applications of dimensional analysis made simple by the use of concepts associated with natural units.

[9] Vacuum breakdown is a fascinating subject first described by Julian Schwinger in 1951 (J. Schwinger, *Phys. Rev.* 82, 664 (1951)). When the electric field is much weaker than $\mathcal{E}_{\rm crit}$ the electron-positron pair must tunnel out to a distance of order $\Delta x$ before they can "become real". So below $\mathcal{E}_{\rm crit}$ pair creation is suppressed by a barrier penetration factor that can be calculated using the WKB method.

# 5 Thinking about fundamental physics with the help of natural units

$c$ and $\hbar$ are the natural units of velocity and action for the world of physics. Their presence indicates the importance of relativity or quantum mechanics respectively. We can learn a great deal about fundamental physics simply by constructing quantities with the right dimensions out of the appropriate ....

## 5.1 Studies in electrodynamics

**The fine structure constant** One number catches the eye in eq. (12): the squared electron charge is proportional to $\hbar c$, so $e^2/\hbar c$ is a pure (dimensionless) number. The units are correct: $[e^2] = m\ell^3 t^{-2} = [\hbar c]$. Let's derive its value: start from the charge of the electron in *cgs* units, $e = 4.803 \times 10^{-10}\,\text{gm}^{1/2}\text{cm}^{3/2}\text{sec}^{-1}$,

$$\begin{aligned} e^2 &= 2.307 \times 10^{-19}\,\text{gm cm}^3\,\text{sec}^{-2} \\ \hbar c &= 3.161 \times 10^{-17}\,\text{gm cm}^3\text{sec}^2, \text{whence} \\ \frac{e^2}{\hbar c} &= 1/137.0359895(61)\ . \end{aligned}$$

$e^2/\hbar c$ is known as the "fine structure constant", because it was first encountered in the study of the fine structure of atomic spectra. It is a fundamental measure of the strength of relativistic (note the $c$), quantum (note the $\hbar$), electrodynamics (note the $e^2$). For example, let's try to estimate the strength of the Coulomb force between an electron and a positron using only length and energy scales intrinsic the the $e^+e^-$ system. The electrostatic potential energy of an electron and positron separated by a distance $r$ is $e^2/r$. The only separation natural to the system is the Compton wavelength, $\lambda\!\!\!^{-}_e = \hbar/m_e c$, and the only scale for measuring energies is the rest energy $m_e c^2$, so the quantity, $(e^2/\lambda\!\!\!^{-}_e)/m_e c^2 = \alpha$ tells us how strong electrodynamic forces are on quantum, relativistic distance scales. Since $\alpha \ll 1$, the effects of electrodynamics are relatively weak and can be treated using the methods of quantum mechanical perturbation theory. Note how different relativistic quantum physics is from classical physics. Classically electrodynamical forces can be made as strong as you like by packing more and more charge into a finite volume. $e^+e^-$ pair creatiion — a quantum relativistic effect — puts a limit on the strength of electromagnetic fields.

**Length scales in electrodynamics** What distance scales can be constructed from $e$, $m_e$, $\hbar$, and $c$? Well, $\lambda\!\!\!^{-}_e = \hbar/m_e c$ is a distance, and $\alpha = e^2/\hbar c$ is dimensionless, so $d_n = \lambda\!\!\!^{-}/\alpha^n, n = 0, \pm 1, \pm 2, ...$ are a series of natural distance scales characterizing electrodynamics. Let's look at a few:

- $n = 0$, $d_0 = \hbar/me = \lambda\!\!\!^{-}_e = 3.861 \times 10^{-11}$ cm: The electron Compton wavelength, clearly the scale at which relativistic quantum effects become important for electrons. Since there is no factor of $e$, it has nothing to do with electrodynamics. An example of its significance: if an electron is confined to $\sim \lambda\!\!\!^{-}_e$ the localization energy from the uncertainty principle is comparable to its rest energy, so electron-positron pairs are created and the system can no longer be considered a single particle state.

- $n = 1$, $d_1 = (\hbar/mc)/(e^2/\hbar c) = \hbar^2/me^2 = a_0 = 0.5291 \times 10^{-8}$ cm: The electron Bohr radius, the scale of the non-relativistic quantum electron bound state. Since there is no factor of $c$, it has no knowledge of relativity. All other combination of $e$, $m_e$, $\hbar$, and $c$ with dimensions of length have factors of $c$, so only $a_0$ can set the scale of the non-relativistic atom. While we're discussing atoms, the binding energy has to be of order $mc^2\alpha^2 = me^4/\hbar^2$ because this is the only energy without a factor of $c$.

- $n = -1$, $d_{-1} = (\hbar/mc) \times (e^2/\hbar c) = e^2/mc^2 = r_e = 2.817 \times 10^{-13}$ cm: The "classical radius" of the electron. The is the size of a sphere of charge whose classical electrostatic potential energy equals the electron's rest energy. This number was much discussed before the discovery of quantum mechanics. We now know that quantum effects become important at much larger distance scales, so $r_e$ is of less interest.

**The root mean square velocity of the electron in the hydrogen atom** Here is quite a non-trivial result about the properties characteristic of electrons in atoms that can be derived from dimensional analysis alone. The electron in an atomic orbital moves with a root mean square velocity $\sqrt{\langle v^2 \rangle}$ that can depend only on $e$, $m_e$, and $\hbar$, and not on $c$ (and, of course, on the dimensionless quantum numbers that specify the orbital).[10] We know

[10] To a good approximation the nucleus of the atom is at rest, so its mass cannot enter the result.

that $[e^2/\hbar] = \ell/t$ because $e^2/\hbar c$ is dimensionless. No other combination of $e$, $m_e$ and $\hbar$ has dimensions of velocity, so $\sqrt{\langle v^2\rangle}/c \simeq e^2/\hbar c = 1/137$. The actual result for hydrogen is $\sqrt{\langle v^2\rangle}/c = N\alpha$, where $N$ is the principal quantum number.

**Corrections to the spectrum of hydrogen** The fine structure of the spectrum of hydrogen comes from the interaction of the orbiting electron's magnetic moment with the magnetic field it sees in its rest frame. Relativity tells us that an electric field seen from moving frame generates a magnetic field of magnitude $|\vec{B}| \sim \frac{v}{c}|\vec{E}|$. For hydrogen $\frac{v}{c} \simeq \alpha$ and the electric field strength can be estimated from the charge and radius of the Bohr atom, $|\vec{E}| \sim \frac{e}{a_0^2} = \frac{m^2e^5}{\hbar^4}$. The problem of finding the natural scale for intrinsic magnetic moments is relegated to Problem 3. The result is $\mu \sim \frac{e\hbar}{mc}$. The interaction energy between the magnetic moment and field is $\Delta E \sim \mu|\vec{B}| \sim me^8/\hbar^4c^2 = \alpha^4mc^2$. Compared to the binding energy of the atom these effects are suppressed by a factor of $\alpha^2 \sim 10^{-4}$.

However there are other effects of the same order that cannot be ignored. In particular, there are relativistic corrections to the kinetic energy. According to Einstein $E = \sqrt{p^2+m^4} = m + \frac{p^2}{2m} + \frac{p^4}{8m^3} + ...$ (note $c$ has been suppressed). Only the $\frac{p^2}{2m}$ contribution is included in the Schrödinger equation. The next term gives a correction of order $\langle v^2\rangle/c^2$, which we now know is of order $\alpha^2$. Since this is the same order as the magnetic interaction energy they have to be treated at the same time. The complete treatment is presented in standard texts.

## 5.2 The quanta of angular momentum, conductance and magnetic flux

One of the most striking features of the quantum world is the appearance of quantization conditions. The quantization of angular momentum is the most famous: since the units of angular momentum are the same as the units of action, $\hbar$ makes a natural appearance in quantum mechanics as the quantum of angular momentum. We no longer think twice that molecular, atomic, nuclear and particle angular momenta are quantized in multiples of the fundamental unit of action, $\hbar$.

In problems involving electrodynamics we can fashion several other *fundamental quanta* out of the basic constants $\hbar$, $c$ and $e$, and we can look for dynamical observables that might be quantized as integer multiples of these quanta. Two examples have played an important role in modern quantum physics, *electrical conductance* and *magnetic flux*.

**Conductance** The conductance is the constant of proportionality between current and voltage in Ohm's Law: $I = \sigma V$. It is the inverse of the resistance and, referring back to eq. (9), is measured in units of *velocity*. Conductance should grow with the strength of the electric charge $e$ and should be independent of the sign of $e$ (since the sign of the charge carriers cannot be determined from Ohm's law). A quantum of conductance that fills all these requirements is $\frac{e^2}{\hbar}$ — the same velocity encountered in the analysis of atoms. So we should expect that for systems where quantum effects dominate the conductance, it comes in multiples of the fundamental *quantum of conductance*, $\sigma_0 = \frac{e^2}{\hbar}$. This quantum has made an appearance in two wonderful recent advances in quantum physics: the Quantum Hall Effect, and Landauer Conductivity in mesoscopic systems. In both cases small systems exhibit quantization of their conductance in units of $\sigma_0$.

The observation of systems with quantized conductance is very surprising if you take your intuition from classical physics. In classical electrodynamics the conductance is a specimen *dependent* quantity obtained by multiplying the *conductivity*, which is a more fundamental property of the material, by an effective cross sectional area and dividing by an effective wire-length. One might expect that conductivity would have some fundamental significance, but not the conductance. However, the units of conductivity are $[\text{conductivity}] = t^{-1}$, and it is not possible to construct a quantity with those units by combining powers of $e$, $\hbar$ and $c$ alone. The physical phenomena that exhibit quantization of conductance do deal directly with the conductance rather than the conductivity because they involve global aspects of the system. Perhaps this unintuitive feature explains why quantization of conductance waited so long to be discovered.

**Magnetic Flux** The magnetic flux is the integral of magnetic field strength over an area, $\Phi = \oint \vec{B} \cdot d\vec{S}$. Referring back again to eq. (16) we see that the natural units of magnetic flux are $m^{1/2}\ell^{3/2}t^{-2}$. These are the same units as the electric charge, $e$. We would not expect $\Phi$ to be quantized in units of $e$ for several reasons: first (and foremost) $\hbar$ does not appear; and second, magnetic effects are typically first order in $\frac{v}{c}$, so we would expect $c$ to appear in the fundamental quantum of flux. Both of these flaws would easily be removed if flux were quantized in units of $\Phi_0 = \frac{e}{\alpha} = \frac{\hbar c}{e}$. This quantum of flux appears in the study of the Aharonov-Bohm Effect and in the study of the motion of a charged particle in a constant magnetic field ("Landau levels").

## 5.3 The Casimir effect

One of the most unusual effects of the quantization of the electromagnetic field is known as the Casimir force. Quantum mechanics requires that the modes of the electromagnetic field experience zero point motion similar to that of the harmonic oscillator. In fact a zero point energy $\frac{1}{2}\hbar\omega$ can be associated with each mode of the electromagnetic field with frequency $\omega$. Normally this energy is invisible because the vacuum is present both before and after we perform any measurement.

The presence of a conductor restricts the allowed modes of the electromagnetic field. Only those modes for which the electric field is normal and the magnetic field is tangential to the conductor are allowed. The modification of the zero point energy generates a force (per unit area) between two conductors, first computed for two infinite parallel plates by Hendryk Casimir in 1948. For perfect conductors there is nothing for the force to depend on other than $\hbar$, $c$, and the separation between the plates, $d$. We can determine the dependence of the force on separation from dimensional consistency alone:

$$\frac{F}{A} \propto \hbar^\alpha c^\beta d^\gamma. \tag{24}$$

The dimensions will be consistent only if $\alpha = \beta = 1$ and $\gamma = -4$, so the Casimir force *goes grows like the fourth power of* $1/d$ *at small distances.* Electrostatic forces grow only like $1/d^2$, so the Casimir effect becomes relatively more important at short distances.

Dimensional analysis cannot tell us the coefficient of proportionality in eq. (24), or even the sign of the force. Casimir found that the force is attractive and the numerical coefficient is small,

$$\left.\frac{F}{A}\right|_{\text{Casimir}} = -\frac{\hbar c \pi^2}{420 d^4} \tag{25}$$

At a distance of one micron ($10^{-4}$ cm) the Casimir force is about 0.3 dynes cm$^{-2}$, small but detectible with modern atomic force microscopes!

# 6 The scale of quantum gravity

So far I have not said anything about gravity. I wanted to leave gravity to the end for two reasons: first, we do not have a description of gravity that is consistent with relativity and quantum mechanics, so we're on shaky ground when we examine *relativistic, quantum* gravity using natural units; and second, the analysis is so powerful that it is the best way to end this discussion!

As you learn in an introductory physics class, gravity — as we encounter it in everyday experience — is described by Newton's Law, which looks deceptively like Coulomb's law in *MKS* units. Compare

$$F_{\text{Newton}} = -G_N \frac{m_1 m_2}{r^2} \quad \Leftrightarrow \quad F_{\text{Coulomb}} = \frac{1}{4\pi\epsilon_0}\frac{Q_1 Q_2}{r^2} \;. \tag{26}$$

where Newton's constant, $G_N = 6.6742(10) \times 10^{-11}\text{m}^3\text{kg}^{-1}\text{sec}^{-2}$, measures the strength of gravitational forces.

The essential difference (aside from the sign) between gravity and electrostatics is that the "gravitational charge" that appears on the right hand side of Newton's Law is just the *inertial mass*, which has already defined by Newton's 2nd law of motion, $F = ma$. This complicates thinking about gravity in natural units.

To get a fresh perspective, suppose gravity had been discovered in the same historical fashion as electrostatics. The propensity for objects to attract one another could be attributed to a "gravitational charge", $\mathcal{Q}$,

$$F_{\text{Gravity}} = -K \frac{\mathcal{Q}_1 \mathcal{Q}_2}{r^2} \tag{27}$$

where the constant, $K$, depends on the units chosen for gravitational charge. After years of experimentation, someone would have realized that gravitational charge is proportional to inertial mass $\mathcal{Q} = \beta m$. [We usually attribute this profound realization to Galileo since it leads to the observation that all objects fall at the same rate in the Earth's gravity.] Then eq. (27) would be replaced by

$$F_{\text{Gravity}} = -K\beta^2 \frac{m_1 m_2}{r^2} \tag{28}$$

which is Newton's law when we define $K\beta^2 = G_N$.

Let's forget, for the moment, about Galileo's great discovery, and return to eq. (27). Following the path of electrostatics, let's define a unit of gravitational charge in the *cgs* system analogous to the *esu*: the "gravitostatic unit" or *gsu* is the amount of matter that produces a gravitational force of 1 dyne at a separation of 1 centimeter. The *cgs* units of the *gsu* are $\text{gm}^{1/2}\text{cm}^{3/2}\text{sec}^{-1}$, just the same as the *cgs* units of the *esu*. The relationship between the *gsu* and the kilogram can be read off from Newton's Law:

$$1\text{gsu} = 3.87080(58)\text{kg} \tag{29}$$

You can easily check that the gravitational force between two masses of 3.87... kilograms separated by 1 centimeter is 1 dyne. In comparison one *esu* is $3.336 \times 10^{-10}$ Coulombs. So the electrostatic force between an everyday quantity of charge (the Coulomb) is much larger than the gravitational force between an everyday quantity of mass (the kilogram). Apparently gravity is much weaker than electromagnetism.

We can pursue the analogy with electrodynamics further by defining a dimensionless measure of the strength of relativistic quantum gravity, $\alpha_G$, analogous to $\alpha = e^2/\hbar c$. We need to know the gravitational charge on the electron, let's call it $g$, in *gsu*. Eq. (29) tells us the relationship between *gsu* and kilograms, so it is easy to convert the electron's mass, $m_e = 9.109 \times 10^{-28}$ gm, to find $g = \sqrt{G_N} m_e = 2.353 \times 10^{-33}$ gsu. Then we find

$$\alpha_G = \frac{g^2}{\hbar c} = 2.759 \times 10^{-46} \ ! \tag{30}$$

This is the gravitational interaction energy of two electrons separated by their Compton wavelengths as a fraction of the electron's rest mass.

The breathtakingly small size of $\alpha_G$ is a measure of the weakness, and therefore the elusiveness of relativistic quantum gravity. One big reason why we have not discovered how to combine quantum mechanics with gravity is that $\alpha_G$ is so small.

The search for a consistent relativistic quantum theory of gravity is one of the great unsolved problems of modern physics, so it is quite interesting to learn at what scales it might be important. To get further information let's write Newton's constant in natural units,

$$\begin{aligned} G_N &= 6.6742(10) \times 10^{-8}\text{cm}^3/\text{gm sec}^2 \\ &= 6.7087(10) \times 10^{-39}\text{GeV}^{-2}\hbar c^5 \end{aligned} \tag{31}$$

When quantum mechanics is important, actions are of order $\hbar$. When relativity is important speeds are of order $c$. Thus $G_N$, written in natural units, gives the energy, length, and time scales of relativistic quantum gravity,

$$\begin{aligned} E_{\text{Planck}} &= \frac{1}{\sqrt{G_N}} \simeq 1.2 \times 10^{19}\text{GeV} \\ \ell_{\text{Planck}} &= \sqrt{G_N} \simeq 1.7 \times 10^{-33}\text{cm} \\ t_{\text{Planck}} &= \sqrt{G_N} \simeq 5 \times 10^{-44}\text{sec} \ , \end{aligned} \tag{32}$$

where all the relations have been written in natural units. All of these are named in honor of Max Planck. Relativistic quantum gravity will be important particles are collided at energies of order $E_{\text{Planck}}$ (our current record is about $10^3$ GeV), or when we probe lengths of order $\ell_{\text{Planck}}$ (our current record is about $10^{-17}$ cm), or when the age of the Universe was about $t_{\text{Planck}}$ (many orders of magnitude earlier than we can now probe). Although a relativistic quantum theory of gravity is one of the great challenges for 21st century physics, the realms where its effects are important are so far from our experimental capacity that we have little to guide us.

# 7 Exercises

1. **cgs and natural units for mechanical quantities**

   Find the cgs units — which must take the form $\text{gm}^a\text{cm}^b\text{sec}^c$ — for each of the following quantities that arise in mechanics. Then express each of them in natural units in the fashion of eq. (16),

   (a) Force

   (b) Surface tension (energy per unit area)

   (c) Number density (number per unit volume)

   (d) Momentum density (momentum per unit volume)

   (e) Energy flux (energy per unit area per unit time)

   (f) Viscosity of a fluid defined by ($\frac{dF_x}{dA} = \eta\frac{\partial v_x}{\partial y}$, where $F_x$ and $v_x$ are the force and velocity in the $x$ direction, which may be functions of $y$).

2. ***cgs* and natural units for electromagnetism**

   There are several slightly different ways for generalizing cgs units to electromagnetism. The differences revolve around factors of $c$ and $4\pi$. We use units where Gauss' Law reads $\vec{\nabla}\cdot E = 4\pi\rho$ and where a factor of $c$ is introduced into the Lorentz force law so that $\vec{E}$ and $\vec{B}$ have the same units. This system is known as "Gaussian units". We prefer to think of it as *cgs* extended to include electromagnetism. The *cgs* units for electric charge and electric field are derived in the lecture notes.

   (a) Using the defining equations for electromagnetic quantities given in eq (7) find both the *cgs* and natural units for $\vec{E}$ and $\vec{B}$, electric and magnetic fields; $\rho$ and $\vec{j}$, charge and current densities; $I$, current; $R$, resistance; $\rho$, resistivity; and $L$, inductance.

   (b) If $\vec{E}$ and $\vec{B}$ are electric and magnetic fields, show that

      i. $\vec{E}^2$ and $\vec{B}^2$ have the units of energy density,

      ii. $\frac{1}{c}\vec{E}\times\vec{B}$ has units of momentum density, and

      iii. $c\vec{E}\times\vec{B}$ has units of energy flux.

3. **Units for magnetic moments**

The magnetic moment ($\vec{\mu}$) of a system determines its interaction energy with a magnetic field,

$$\Delta E = \vec{\mu} \cdot \vec{B}. \tag{33}$$

(a) What are the *cgs* units for magnetic moments?

(b) What are the natural units for magnetic moments?

(c) A particle's magnetic moment is linearly proportional to its electric charge, $e$, and to its spin, $\vec{S}$. Write an expression for a particle's magnetic moment in terms of its mass ($m$) and the fundamental constants $\hbar$ and $c$. Call the dimensionless constant of proportionality $g$.

(d) For an electron, $g_e \simeq 2$ and its spin is quantized to $\pm\hbar/2$. What is the interaction energy for an electron when its spin is parallel to a magnetic field of $10^5$ gauss?

4. **The Casimir Polder Force**

Before his work on parallel plates, Casimir together with X. Polder studied the quantum mechanical force between two polarizable molecules. The form of this force can be predicted from dimensional considerations.

(a) The *static polarizability*, $\alpha$, of a system is defined by the equation,

$$\vec{p} = \alpha \vec{E} \tag{34}$$

where $\vec{p}$ is the electric dipole moment (with dimensions charge$\times$ distance) and $\vec{E}$ is a constant electric field. Show that the dimensions of $\alpha$ are $\ell^3$.

(b) In quantum mechanics molecule-1 can fluctuate to a polarized state, thereby producing an electric field that polarizes molecule-2 (or *vica versa*). The two then attract one another until they fluctuate back to an unpolarized state. Casimir and Polder used relativistic quantum mechanics and found a force between the two molecules proportional to the product of their static polarizabilities and not depending on any of their other properties,

$$F = -K\alpha_1\alpha_2/r^p \tag{35}$$

(The minus sign denotes attraction). Determine the parameter $p$ by demanding consistency of dimensions. [Hint: By hypothesis $K$ depends only on $\hbar$ and $c$.]

(c) If the same calculation is done using *non-relativistic* quantum mechanics one also finds a power law. Can this force be proportional to the product of static polarizabilites as well? If so, what is the power law? If not explain why not.

5. **Thermodynamics in natural units and applications**

The concept of temperature can be integrated into the natural unit system by remembering that $kT$ is an energy, where $k = 8.617343(15) \times 10^{-5}\text{eV}/^\circ K$, is Boltzmann's constant. So *a temperature is given by the equivalent energy* or, in other words, "Boltzmann's constant is set equal to unity". The familiar statement that room temperature is equivalent to about 1/40 eV, made more precise,

$$300^\circ\text{K} = [38.681684(68)]^{-1}\text{eV} \ , \tag{36}$$

is a useful conversion factor.

(a) To what temperature in °K does the mass of the electron correspond?

(b) The lowest temperatures achieved in recent years are a few micro-Kelvin. What is $1^\circ\mu$K in eV?

(c) Entropy is defined in thermodynamics by $dS = \frac{dQ}{T}|_{\text{reversible}}$, or in statistical mechanics by $S = k \ln \Gamma$ (where $dQ_{\text{reversible}}$ is the amount of heat added reversibly to a system, and $\Gamma$ is the number of configurations of a system with identical macroscopic properties). What are the units of entropy in the natural unit system?

(d) Recently D. T. Son has suggested that there is a fundamental bound on the *viscosity* ($\eta$) of a system in terms of its entropy density ($\sigma$ — the entropy per unit volume). Son's conjecture is

$$\eta \geq \frac{\hbar}{4\pi}\sigma \tag{37}$$

Show that Son's conjucture is dimensionally consistent. The viscosity of water at standard temperature and pressure is given by

$0.89 \times 10^{-3}$ Pascal-seconds, and its entropy per unit volume is $2.8 \times 10^{-23}$ cm$^{-3}$. Calculate the ratio $\eta/\sigma$ and show that the Son bound is satified. [Note: You will need to convert Pascal-seconds to *cgs* units.]

(e) One of the very few results from relativistic quantum gravity is Hawking's conjecture that black holes radiate thermal ("black body") radiation characterized by a temperature proportional to their their surface area: $T_{\rm Hawking} = CA$. [The surface area of a black hole is $4\pi$ times the square of its Schwartzschild radius.] Recognizing that $C$ depends on *relavistic* ($c$), *quantum* ($\hbar$), *gravity* ($G_N$), write

$$C = N\hbar^{p_1}c^{p_2}G_N^{p_3}, \tag{38}$$

where $N$ is a pure number and determine $p_1$, $p_2$, and $p_3$ by demanding the consistency of dimensions in $T_{\rm Hawking} = CA$.

Hawking found $N = 1/4$. What is the Hawking temperature (in °K) of a black hole with a radius of 2 kilometers (which is approximately the Schwartzschild radius of the Sun)?

6. **The Bag Model of the Proton**

The proton and neutron ("nucleons" for short) are made of three almost massless quarks. The mass of the nucleon is approximately 940 MeV. The quarks are confined to the interior of nucleons because it takes work to "open up" a region of space, a "bag", in which they can be present. A very simple model of the nucleon treats it as a spherical bag of radius $R$. The energy includes only two terms: a) the work done to create the bag, and b) the zero point energy of the confined quarks. The work that must be done to create the bag is parameterized by an energy per unit volume, $B$, known as the bag constant. So it takes energy $E(\text{bag}) = BV$ to create a nucleon bag of volume $V$. The three quarks have kinetic energy $E(\text{kinetic}) = 3\langle|\vec{p}|\rangle$ (remember $c = 1$) and $\langle|\vec{p}|\rangle$ can be estimated from the uncertainty principle: $\langle|\vec{p}|\rangle \approx 1/R$ (remember $\hbar = 1$), where $R$ is the radius of the nucleon's bag. So the total rest energy (*ie.* its rest mass) of a three quark bag as a function of $R$ is

$$\begin{aligned} E(R) &= E(\text{kinetic}) + E(\text{bag}) \\ &= \frac{3}{R} + \frac{4\pi}{3} B R^3 \end{aligned} \tag{39}$$

The radius $R$ will adjust dynamically to minimize the total energy.

(a) Restore the factors of $\hbar$ and $c$ to eq. (39) to make it dimensionally correct. Then set $\hbar = c = 1$ for the rest of the problem.

(b) Find the radius $R_0$ which minimizes the bag's rest energy. $R_0$ will be a function of $B$. Find the mass of the nucleon as a function of $B$ by substituting $R = R_0$ in eq. (39).

(c) Given the mass of the nucleon, $M = 940$ MeV, find the numerical value of $B$. What are its units in the natural unit system?

(d) $B$ has the units of pressure. Convert the answer from part (c) into atmospheres. Don't be surprised by the huge value of your answer: This is the pressure exerted by the vacuum on quarks!

(e) Compute the radius $R_0$ in centimeters. The observed radius of the nucleon is about $1.0 \times 10^{-13}$ cm. Given that we made several very crude estimates, for example we chose $\langle|\vec{p}|\rangle \approx 1/R$ instead of $\langle|\vec{p}|\rangle \approx \pi/R$, what do you think of the accuracy of your answer?

7. **The Discovery of "Wierdness"**

In the year 2010 an experimenter discovers a new force between electrons. After many experimental studies, she determines that it falls like the 3/2 power of distance,

$$\vec{F}_{12} = \frac{w_1 w_2}{r^{\frac{3}{2}}} \hat{r}_{12} \tag{40}$$

$w_k$ measures the strength that this new force couples to particle $k$. Because this new force is wierd, the constants $w_k$ become known as "wierdness". After many years of hard work, the wierdness of the electron is measured to be

$$w_e = 1.18 \pm 0.05 \times 10^{-5} \text{w.s.u.}, \tag{41}$$

where the w.s.u. is the cgs unit of wierdness.

(a) What are the cgs units of wierdness in terms of $\text{gm}^a\text{cm}^b\text{sec}^c$?

(b) Using the basic constants of nature, $\hbar$ and $c$, find a measure of the energy scale where the relativistic quantum wierdness of the electron will be important. Express your answer in the form – $E_{\text{wierdness}} = w_e^\alpha \hbar^\beta c^\gamma$.

(c) Evaluate $E_{\text{wierdness}}$ in MeV.

8. **Supernatural units!**

Theorists who study relativistic quantum gravity use an extended version of natural units where they take $\hbar = c = G_N = 1$. Let's call them "supernatural units".

(a) Explain why in this system of units all physical quantities are *pure (dimensionless) numbers*!

(b) Write a new set of conversion relations giving 1 sec, 1 gm, and 1 cm in terms of $G_N$, $\hbar$, and $c$, analagous to eq. (11)

(c) What is the mass $M = 1$ in supernatural units? What is the length, $L = 1$? [Hint: you will find eq. (31) valuable.]

(d) The age of the Universe is approximately $10^{10}$ years. What is this in supernatural units?

(e) Recently astrophysicists have discovered that the universe is filled with a substance, called "dark energy" which behaves like a negative *bag constant* (see Problem 6). It causes the Universe to expand just like the bag constant tries to make a proton contract. The dark energy density is approximately xx erg/cm$^3$. Express this in supernatural units.

The astounding size of the dimensionless numbers describing the age of the Universe and the dark energy density in supernatural units constitute one of the principal puzzles of modern physics.