Quantum Physics III (8.06) Spring 2008
Assignment 1

Feb 5, 2008 **Due Tuesday Feb 12, 2008**

- Please remember to put **your name and section time** at the top of your paper.

Readings

The reading assignment for the first three lectures of 8.06 is:

- Supplementary notes on Natural Units by Prof. Jaffe (posted on the web).
- Griffiths, Ch. 5.3
- Cohen-Tannoudji, Ch. XI Complement F.

Note:

I will not discuss the Supplementary notes on Natural Units by Prof. Jaffe in lecture. However, it is important for you to read through it carefully. The primary motivation for the notes is that physicists think in natural units, and it's about time that you start doing so too. The notes also provide many beautiful examples of dimensional analysis.

Problems 1-3 in this pset deal with natural units and dimensional analysis. You should find Supplementary notes on Natural Units very helpful in doing them. Using natural units also simplifies the calculation in other problems.

Useful Facts

- Mass of the Sun: $M_{\rm sun} = 1.99 \times 10^{33}$ grams
- The electron mass: $m_e = 511\,{\rm keV}/c^2$
- $$\begin{aligned} \hbar &= 1.05 \times 10^{-27}\,{\rm gm\ cm^2 sec^{-1}} \\ c &= 2.998 \times 10^{10}\,{\rm cm\ sec^{-1}} \\ {\rm eV} &= 1.6 \times 10^{-12}\,{\rm gm\ cm^2 sec^{-2}} \quad . \end{aligned} \tag{1}$$
- $$\begin{aligned} 1{\rm sec} &= 1.52 \times 10^{15} \hbar\,{\rm eV}^{-1} \\ 1{\rm cm} &= 5.07 \times 10^{4} \hbar c\,{\rm eV}^{-1} \\ 1{\rm gm} &= 5.61 \times 10^{32} {\rm eV}\,c^{-2} \end{aligned} \tag{2}$$

- $\hbar c = 197\,\text{MeV}\,\text{fm}$
- $e^2 = \frac{1}{137}\hbar c$ where e is the electron charge.

Problem Set 1

1. **Natural Units (14 points)**

   In cgs unit system all physical quantities are expressed in terms fundamental units of

$$\begin{aligned} \text{length} &: \quad \text{cm} && (3)\\ \text{mass} &: \quad \text{gm (gram)} && (4)\\ \text{time} &: \quad \text{sec (second)} && (5) \end{aligned}$$

   cgs system is natural to use when describing the macroscopic world. However, it is not convenient when dealing with systems with very small sizes where relativity and quantum mechanics are important. Relativity introduces a fundamental constant, the speed of light c, and relativistic effects are controlled by the ratio velocity/c. Quantum mechanics introduces a fundamental constant $\hbar$, which characterizes quantum effects. To make these effects manifest, it is convenient to use a unit system with c and $\hbar$ as fundamental units. Natural units use the following quantities as basic units

$$\begin{aligned} \text{velocity} &: \quad c \\ \text{action} &: \quad \hbar \\ \text{energy} &: \quad eV && (6) \end{aligned}$$

   where eV denotes electron-volt. In particle and astrophysics, natural units are universally used. It is important you get used to them.

   (a) (4 points) Consider some physical quantity Q, which has dimension

$$[Q] = [\text{gm}]^a[\text{cm}]^b[\text{sec}]^c \tag{7}$$

   in cgs units. In natural units Q has dimension

$$[Q] = [c]^\alpha[\hbar]^\beta[eV]^\gamma \tag{8}$$

   Find α, β, γ in terms of a, b, c.

(b) (10 points) Express the dimensions of the following physical quantities in terms of natural units, e.g. [mass] = eVc^{-2},

$$[\text{force}], \quad [\text{pressure}], \quad [\text{conductivity}],$$
$$[\text{magnetic moment}], \quad [\text{viscosity}]$$

Note: the viscosity η of a fluid is defined by

$$\frac{dF_x}{dA} = \eta \frac{dv_x}{dy} \tag{9}$$

where $\frac{dF_x}{dA}$ and v_x are the force per unit area and velocity in the x direction, which may be functions of y.

2. **Planck scales** (15 points)

Natural units (6) are in fact not the most natural ones since while $\hbar$ and c are fundamental constants in nature, eV is not. In nature there is another fundamental constant G_N if we include gravity

$$F_{\text{Newton}} = -G_N \frac{m_1 m_2}{r^2} \tag{10}$$

with Newton constant G_N given by

$$G_N = 6.67 \times 10^{-8}\, cm^3/(gm\ s^2)\ . \tag{11}$$

The most natural units in nature can then be chosen to be (sometimes called supernatural units or simply Planck units)

$$\hbar, \qquad c, \qquad G_N \tag{12}$$

(a) (3 points) Write down combinations of (12) which have dimension of mass, length and time respectively. These quantities are called Planck mass M_{pl}, Planck length l_{pl} and Planck time t_{pl} respectively.

(b) (5 points) Find the value of M_{pl} in both cgs and natural units. For a particle of mass M_{pl}, what are the physical interpretation of l_{pl} and t_{pl}? Find the values of l_{pl} and t_{pl} in cgs units.

(c) (4 points) Find the ratio

$$\frac{E_{grav}}{m_e c^2} \tag{13}$$

where E_{grav} is the gravitational energy between two electrons separated by a distance equal to the Compton wave length of an electron. $m_e c^2$ is the static energy of an electron. This ratio characterizes the strength of gravity between two electrons. What can you conclude from this calculation?

(d) (3 points) Consider a particle of mass M_{pl}. Find the ratio

$$\frac{E_{grav}}{M_{pl}c^2} \tag{14}$$

where E_{grav} is the gravitational energy between two such particles separated by a distance equal to their own Compton wave length. This calculation tells you that M_{pl} can be interpreted as the mass scale that *quantum gravitational effects* become important.

[**Note:** The units (12) was first introduced by Planck in 1899 shortly after he introduced $\hbar$. Planck further suggested: l_{pl}, M_{pl}, t_{pl} are

"the units of length, mass, time that would, independently of special bodies and substances, necessarily retain their significance for all times and all cultures, even extraterrestrial and extrahuman ones, and which may therefore be designated as natural units of measure." (Planck 1899)

However, Planck did not understand the physical meaning of these scales. The smallness of l_{pl} and t_{pl} (as you would have found out from (b)) baffled him. The physical interpretation of M_{pl} (and thus l_{pl} and t_{pl}), based on arguments very similar to (d), was only realized in 1950's by Landau, Klein, Wheeler.]

3. **The accelerating universe (5 points)**

(a) (3 points) One of the biggest recent discoveries in science is that the expansion of the universe is accelerating. This can be described, but not really understood, by reintroducing what Einstein called the cosmological constant Λ into the equations of general relativity[1]. Λ describes a uniform energy density permeated over the whole universe and thus has the dimension of the energy density. What is the most natural scale for Λ in a theory of quantum gravity? (hint: recall problem 2(e).)

(b) (2 points) The measured value of Λ — in other words, the Λ that must be introduced in order to parameterize the recently observed accelerating expansion of the universe — is $\Lambda \approx 2000$ eV/cm^3. What is the ratio of the observed Λ to the answer you find in (a)? You should find this ratio extremely small. Understanding the smallness of Λ is one of the most outstanding challenges in theoretical physics.

[If you are interested in reading more about the cosmological constant, you can find an article from Scientific American at the following URL:

http://atropos.as.arizona.edu/aiz/teaching/a204/darkmat/SciAm99b.pdf]

4. **Fermi energy, velocity and temperature of copper (8 points)**

Do Griffiths, Problem 5.16.

[1]Einstein called this his biggest blunder.

5. **Free fermion gas in a two dimensional well (6 points)**

   Do Griffiths, Problem 5.34.

6. **White dwarfs and Neutron stars (6 points)**

   Note: You can directly use formulae derived in lecture for this problem as far as you state clearly the meaning of those formulae.

   (a) (3 points) Consider a white dwarf star of the same mass as the Sun. Assume that the star is mainly made of Carbon. What is the radius of star? Note that the Sun has a mass of $M_{sun} = 2 \times 10^{33} g$ and a radius of $R_{sun} = 7 \times 10^5 km$. Find the ratio of the mass density of the white dwarf and the Sun.

   (b) (3 points) In a neutron star, the neutron degeneracy pressure stabilizes the collapse. Calculate the radius of a neutron star with the mass of the Sun. You can assume that the star only consists of neutrons and the neutron gas is free[2]. Find the (neutron) Fermi energy and compare it to the rest energy of a neutron.

7. **"Free" electron gas** (6 points)

   In our discussion of electron gas in alkali metal and white dwarf, we have made the assumption that Coulomb interactions between electrons are not important.

   (a) (4 points) Express in words or simple equations, how would you formulate a criterion to check the assumption.

   (b) (2 points) For a white dwarf of mass of the Sun, use your criterion to check whether the assumption is valid.

[2] This is not a very good approximation to the realistic situation in which nuclear interactions between neutrons are important.