Quantum Physics III (8.06) Spring 2008
Assignment 10

May 5, 2008

- **You do not need to hand this pset in.**
- The solutions will be provided after Friday May 9th.
- **Your FINAL EXAM is MONDAY MAY 19, 1:30PM-4:30PM, in JOHNSON ICE RINK.**
- Sample final exams are now available on the web (Solutions will be posted next week).
- Additional office hours in the final week of class will soon be announced.

Readings

- Griffiths Chapter 11
- Prof. Jaffe's notes on scattering

Problem Set 10

Even though you do not need to hand them in, please do them carefully. This pset also serves as a study guide to help you prepare for the scattering part of the final exam.

Solutions to all problems will be provided.

1. **Scattering from a Reflectionless Potential (10 points)**

   Consider a particle of mass m moving in one dimension under the influence of the potential

   $$V(x) = -\frac{\hbar^2 a^2}{m}\text{sech}^2(ax) \ .$$

   (a) This potential has a normalizable bound state with wave function $\psi_0(x) \propto \text{sech}(ax)$. What is its energy?

   (b) Show that

   $$\psi(x) = \left(\frac{k}{a} + i\tanh(ax)\right)\exp ikx$$

   is a solution to the same problem with energy $E = \hbar^2 k^2/2m$.

(c) Now consider scattering of a particle with energy E from $V(x)$. Explain (should be brief) that the solution of part (b) satisfies the boundary conditions appropriate for this scattering problem, with the particle incident from the left. Use this solution to show that the reflection coefficient is zero, and to determine the transmission coefficient $T(E)$. Show $|T(E)| = 1$.

(d) Show that $T(E)$ has a pole at the energy of the bound state.

2. **Simple Properties of Cross Sections (15 points)**

Scattering in three dimensions introduces some new concepts: cross sections, scattering amplitudes, solid angle, to name a few. This problem should help you understand the basics.

Consider a scattering wave function in three dimensions parametrized by a function $f(\theta, \phi)$:

$$\psi(r, \theta, \phi) = e^{ikz} + \frac{f(\theta, \phi)}{r} e^{ikr} \ .$$

The first term describes an incident plane wave. The second term describes the scattered flux, scattered off some potential localized in the vicinity of $r = 0$. This scattering wave function is only valid at large r. $f(\theta, \phi)$, which parametrizes the scattered flux, is called the scattering amplitude.

The probability flux for the Schrödinger equation is given by

$$\vec{S} = \frac{\hbar}{2mi} \left(\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^* \right) \ .$$

(a) Compute the incident flux. Calculate the scattered flux for $\theta \neq 0$. [Note: when calculating the scattered flux, keep only the dominant term at large r.]

(b) Define the cross section per unit solid angle by

$$\frac{d\sigma}{d\Omega} d\Omega = \lim_{r \to \infty} \frac{\vec{S}_{\text{scattered}} \cdot \hat{r}}{|S_{\text{incident}}|} dA \ ,$$

where S_{incident} is the incident flux, $S_{\text{scattered}}$ is the scattered flux, and dA is a small element of area, $dA = r^2 d\Omega$, on a distant sphere.
Show that

$$\frac{d\sigma}{d\Omega} = |f(\theta, \phi)|^2 \ .$$

[Note that Griffiths denotes $d\sigma/d\Omega$ by the symbol $D(\theta, \phi)$. This notation is unconventional, but it is helpful in reminding one what $d\sigma/d\Omega$ depends on.]

(c) From considerations of flux conservation, derive the optical theorem:

$$\sigma_{\text{elastic}} \equiv \int d\Omega |f(\theta,\phi)|^2 = \frac{4\pi}{k}\text{Im}f(\theta=0)\ .$$

Hint: Prof. Jaffe's notes are helpful.

3. **Born Approximation for Scattering From Yukawa and Coulomb Potentials, plus a Practical Example of the Latter (15 points)**

Make sure you are aware of Griffiths' Examples 11.5 and 11.6 on page 415 as you do this problem. He has done some of the work for you.

Consider a Yukawa potential

$$V(r) = \beta\frac{\exp(-\mu r)}{r}$$

where β and μ are constants.

(a) Evaluate the scattering amplitude, the differential cross section $d\sigma/d\Omega$, and the total cross section in the first Born approximation. Express your answer for the total cross section as a function of the energy E.

(b) Take $\beta = Q_1Q_2$ and $\mu = 0$, and show that the differential cross section you obtain for scattering off a Coulomb potential is the same as the classical Rutherford result. Use this differential cross section in part (d) below.

(c) Differential cross sections are what physicists actually use to calculate the rate at which scattered particles will enter their detectors. The number of particles scattered into solid angle $d\Omega$ per second by a single scatterer is given by

$$\frac{d^2N}{dtd\Omega} = \frac{d\sigma}{d\Omega} \times \frac{d^2N}{dtdA}$$

where $d^2N/dtdA$ is the incident flux in units of particles per second per unit area, ie per unit cross sectional area transverse to the beam. Consider a uniform beam of dN/dt particles per second with a cross sectional area A. This beam strikes a target with density n (n is the number of scattering sites per unit volume) and thickness t.

Give an expression for the number of particles scattered into a detector with angular size $d\Omega$ per unit time.

Show that your result is independent of the cross sectional area of the beam even if the beam is not uniform across this area. [Note that this is important, because it is typically easy for an experimenter to measure dN/dt but hard for her to measure either A or the uniformity of the beam across the cross sectional area.]

(d) Consider a beam of alpha particles ($Q_1 = 2e$) with kinetic energy 8 MeV scattering from a gold foil. Suppose that the beam corresponds to a current of 1 nA. [It is conventional to use MKS units for beam currents. 1 nA is 10^{-9} Amperes, meaning 10^{-9} Coulombs of charge per second. Each alpha particle has charge $2e$, where $e = 1.6 \times 10^{-19}$ Coulombs.] Suppose the gold foil is 1 micron thick. You may assume the alpha particles scatter only off nuclei, not off electrons. You may also assume that each alpha particle scatters only once. You will need to look up the density of gold and the nuclear charge of gold (Q_2). How many alpha particles per second do you expect to be scattered into a detector which occupies a cone of angular extent ($d\theta = 10^{-2}$ radians, $d\phi = 10^{-2}$ radians) centered at $\theta = \pi/2$?

(e) Suppose you now move the detector around (keeping it at the same distance from the target and thus keeping the solid angle subtended by the detector the same.) How does the number of particles per second seen in the detector depend on the angular location of the detector, θ? What is the number of particles per second seen in the detector for $\theta = 10°$, $\theta = 45°$ $\theta = 135°$, $\theta = 170°$?

4. **The Size of Nuclei (10 points)**

In lecture we derived an expression for the scattering amplitude in the Born approximation for the elastic scattering of a particle of mass m and charge -$|e|$ from a charge distribution $|e|\rho(\vec{r})$:

$$f(\vec{q}) = \frac{2me^2}{\hbar^2 q^2} \int d^3r e^{-i\vec{q}\cdot\vec{r}} \rho(\vec{r}) \ .$$

Recall that $\vec{q} = \vec{k}' - \vec{k}$ is the momentum transferred to the scattered particle in the collision. For elastic scattering, $q \equiv |\vec{q}| = 2|\vec{k}| \sin(\theta/2)$. If the electrons used in a scattering experiment are relativistic, $k \simeq E/c$.

(a) The charge distribution of a nucleus is *not* localized at a mathematical point. f is therefore not exactly that for Rutherford scattering. The charge distribution is roughly constant out to a radius R and then drops rapidly to zero. A simple model is:

$$\rho(\vec{r}) = \frac{3Z}{4\pi R^3} \quad \text{for } r \leq R$$

and $\rho = 0$ for $r > R$. Calculate the cross section for electron scattering from such a nucleus as a function of q^2.

(b) The ratio of the actual amplitude for scattering from a point nucleus is called the "form factor". Sketch the form factor as a function of qR.

The form factor tells us about the "shape" of the charge distribution in a nucleus, and thus tells us how the protons within a nucleus are arranged. In our simple model, the form factor tells us the value of R. If nuclei had precisely the shape we have used in our simple model, experimenters would measure a form factor with precisely the functional form you have calculated, and would then do a fit to obtain a measurement of R, the radius of the nucleus.

(c) For relativistic electrons with energy E, if you are able to count the scattered electrons at a variety of angles, ranging from θ close to zero to θ close to π, what range of q can you access? If you use electrons with $E \ll 1/R$, show that you will not be able to make an accurate determination of R. You will not be able to "resolve" the fact that scattering off a nucleus differs from Rutherford scattering. The values of R for nuclei are around $(2-7) \times 10^{-13}$ cm. Roughly how large an electron energy do you need in order to do a reasonable measurement of R?

First aside: The above problem uses a simple model, but it is not all that far from the real thing.

Second aside: The next step in the process of unveiling the structure of matter on smaller and smaller length scales was the discovery that the protons and neutrons that make up a nucleus have substructure. Electron beams with energies appropriate for studying nuclear structure (ie the distribution of protons within a nucleus, which you've been analyzing in this problem) cannot resolve the substructure of a proton. Thus, the discovery of the quark structure of the proton had to wait until the construction of the SLAC linear accelerator, which began accelerating electrons to 21 GeV in the late 1960's. In 1967, Jerome Friedman, Henry Kendall and Richard Taylor began the series of experiments in which quarks were discovered. When an 21 GeV electron scatters at large angles off a quark in a proton, the proton does not remain intact. This means that the description of these experiments requires an understanding of *inelastic* scattering. In an inelastic collision, the scattered electron's momentum changes by $\vec{q}$, and its energy also changes.

5. **The Born Approximation in One Dimension (15 points)**

(a) Do Griffiths Problem 11.16.

[You need not derive the Green's function as Griffiths does in his text. Rather, it is sufficient for you to take the answer Griffiths gives, and show that it is indeed the integral equation for the one dimensional Schrödinger equation.]

(b) Do Griffiths Problem 11.17.

(c) Do Griffiths Problem 11.18.

6. **Scattering from a Small Crystal (10 points)**

We want to investigate the structure of a crystal by scattering particles from it. The particle sees the potential

$$V(\vec{x}) = \sum_i v(\vec{x} - \vec{X}_i)$$

where the $\vec{X}_i$ are the position vectors of the scattering atoms and $v(\vec{x})$ is the scattering potential of a single atom. Assume that v is weak enough that we can use the Born approximation for the whole crystal, ie for V.

(a) Express the differential cross section as the product of two factors, one of which depends on v and the other on the structure of the crystal, ie the set of points $\vec{X}_i$. Both factors will depend on the momentum transfer $\vec{q}$.

(b) Briefly, compare to whatever you know about Bragg scattering.