Quantum Physics III (8.06) Spring 2008
Assignment 5

March 4, 2008 **Due Tuesday March 11, 2008**

- Please remember to put your name **and section time** at the top of your paper.
- Remember that your **midterm** will be on **Thursday March 20**, in class.

Readings

The reading assignment for this problem set and part of the next one is:

- Griffiths all of Chapter 6.
- Cohen-Tannoudji Chapter XI including Complements A-D.

Problem Set 5

1. **The Aharonov-Bohm Effect on Energy Eigenvalues (15 points)**

   You have seen the "standard presentation" of the Aharonov-Bohm effect in lecture. The standard presentation has its advantages, and in particular is more general than the presentation you will work through in this problem. However, students often come away from the standard presentation of the Aharonov-Bohm effect thinking that the only way to detect this effect is to do an interference experiment. This is not true, and the purpose of this problem is to disabuse you of this misimpression before you form it.

   As Griffiths explains on pages 385-387 (344-345 in 1st Ed.), the Aharonov-Bohm effect modifies the energy eigenvalues of suitably chosen quantum mechanical systems. In this problem, we work through the same physical example that Griffiths uses.

   Imagine a particle constrained to move on a circle of radius b (a bead on a wire ring, if you like.) Along the axis of the circle runs a solenoid of radius $a < b$, carrying a magnetic field $\vec{B} = (0, 0, B_0)$. The field inside the solenoid is uniform and the field outside the solenoid is zero. The setup is depicted in Griffiths' Fig. 10.10. (10.12 in 1st Ed.)

   (a) (2 points) Construct a vector potential $\vec{A}$ which describes the magnetic field (both inside and outside the solenoid) and which has the form $A_r = A_z = 0$ and $A_\phi = \alpha(r)$ for some function $\alpha(r)$. We are using cylindrical coordinates z, r, ϕ.

(b) (2 points) Now consider the motion of a "bead on a ring": write the Schrödinger equation for the particle constrained to move on the circle $r = b$, using the $\vec{A}$ you found in (a). Hint: the answer is given in Griffiths.

(c) (3 points) Solve the Schrodinger equation of (b) and find the energy eigenvalues and eigenstates.

(d) (4 points) Plot the energy eigenvalues as a function of the enclosed flux, Φ. Show that the energy eigenvalues are periodic functions of Φ with period Φ_0, where you must determine Φ_0. For what values of Φ does the enclosed magnetic field have no effect on the spectrum of a particle on a ring? Show that the Aharonov-Bohm effect can only be used to determine the fractional part of Φ/Φ_0.

(e) (4 points) Suppose we introduce a defect on the ring at $\phi = 0$, which can trap the particle, i.e. in addition to the states you worked out above, there now exist trapped states in which the wave function of the particle is localized around $\phi = 0$. For simplicity, assume the trapped state wave functions vanish outside an interval $(-\phi_0, \phi_0)$ for some $\phi_0 < \pi$. Show that the energy of a trapped state does NOT depend on the existence of the solenoid.

[Hint: find a gauge in which the vector potential vanishes identically in the region where trapped state wave functions are supported. You should also explain why the same argument does not apply to states of part (c).]

[Moral of problem: even though the bead on a ring is in a region in which $\vec{B} = 0$, the presence of a nonzero $\vec{A}$ affects the energy eigenvalues of extended states (those states whose wave functions cover the whole circle). $\vec{A}$, however, does not affect the energies of localized states. This is the counterpart for the energy spectrum of a similar statement for an interference experiment: the interference pattern is shifted *if and only if* the relevant paths enclose the solenoid.]

2. **Quantization of Hall conductivity in Integer and Fractional Quantum Hall Effects (5 points)**

In lecture, we gave an explanation for the appearance of plateaus of Hall conductivity in Integer Quantum Hall Effects. We explained that for a material with impurities, one expects the density of states of the system to behave as in figure 1. Since only extended states can conduct a Hall current, there is then a plateau for each filled band of extended states. Note that this argument does not depend on the detailed structure of the density of states. For example, as far as there exist extended states, the number of extended states in a band is not important.

Experiments also observed that at the plateaus the values of σ_H are exactly integral multiples of of $\frac{e^2}{h}$, which coincide with the values for a free electron gas at integer fillings. Note that due to impurities one expects the number of extended states for a real material to be much smaller than that in a filled Landau level of a free electron

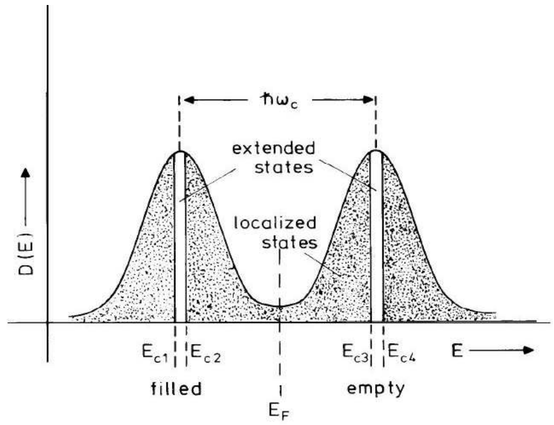

Figure 1: Density of states.

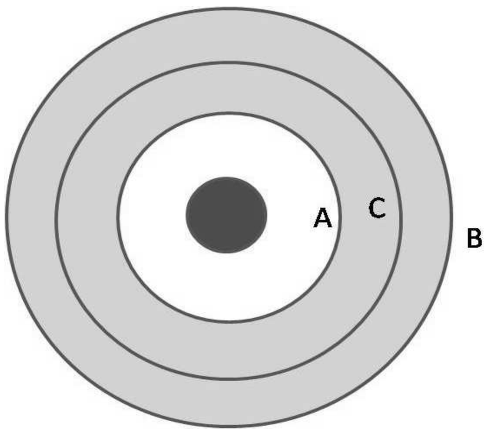

Figure 2: A material of annulus shape with a solenoid inserted in the hole.

gas[1]. Thus it is puzzling why each filled band of extended states in a real material contributes exactly the same amount to σ_H as a filled Landau level of a free electron gas. More mysteriously, the number of extended states in general depends on the type of material, the purity of sample etc, yet the quantization of σ_H appears to be universal.

The exactness and universality of the quantization of Hall conductance suggests that the effect should have a simple explanation, again independent of the detailed structure of the density of states. In this problem I walk you through a simple, but very beautiful argument due to Laughlin and Halperin[2]. The argument has three parts, presented in (a)-(c) below. Only in (b) will there be a question for you. In (d) I also give a brief introduction to Fractional Quantum Hall Effects. This problem is a bona fide part of this problem set, and "counts" in that sense. However, it can also be considered as an extended reading material, which will not appear in the midterm or final.

Consider a material in the shape of an annulus as shown in figure 2 in the presence of a magnetic field B in z-direction (pointing out of the paper). Suppose that the Hall conductance of the material is given by

$$\sigma_H = \nu \frac{e^2}{h} \tag{1}$$

for some value of ν. For the moment we do not need to assume that ν is an integer.

(a) Now insert a solenoid into the hole of the annulus as shown in the figure and very slowly increase the flux of the solenoid from zero to $\Phi_0 = \frac{hc}{e}$. We note two very important aspects of this process:

   i. From our study of the Aharonov-Bohm effect, we can conclude that the Hamiltonian of the system is the same at the beginning and the end of process. Zero flux and flux Φ_0 give the same Hamiltonian for the electrons in the sample.

   ii. If the system is in a non-degenerate ground state, i.e. if the state is separated from other states by a nonzero energy gap, then the system goes back to exactly the same state at the end of the process, provided the changing of the magnetic flux is sufficiently slow. The statement is intuitively obvious, since for a very slow process, there is not enough energy to excite the system. This statement is a consequence of the adiabatic theorem, which we will prove rigorously in the second half of this semester. The theorem also implies that an electron in an extended state cannot jump into a localized state during the process.

[1] This can also be deduced by the fact that the jump between neighboring plateaus is sharp.

[2] The original papers are: R. Laughlin, Phys. Rev. B. **23**, 5632 (1981) and B. Halperin, Phys. Rev. B. **25**, 2185 (1982). These papers are short (Laughlin'paper has only 2 pages) and readable.

(b) Consider a circular path C (as indicated in figure 2), within the sample, encircling the solenoid. Use (1) to determine how much charge (call it Q) crosses this circular path as the flux in the solenoid increases from 0 to Φ_0. You should find that $Q = \nu e$ (do not worry about its sign).

(c) Suppose the Fermi surface for the material lies between two bands of extended states as indicated in the figure 1 by E_F, i.e. the lower band of extended states is fully filled. Then as we slowly increase the flux from zero to Φ_0, the system must remain in the same filled state, since it takes a finite amount of energy to excite an extended electron to a higher extended state. This statement has a caveat. For electrons living near the two edges of the annulus (i.e. A and B as indicated in the figure 2), the story is more complicated due to boundary effect and those electrons could in principle be excited. The result you found in (b) indicates that there are ν electrons crossing each closed paths enclosing the solenoid. Since in the bulk of the material, the electronic states are unchanged, the only way to reconcile with the result of (b) is that there are ν electrons which moved from one edge to the other. Since we cannot have fractional electrons, one concludes that ν must be an integer[3]. This establishes the quantization of Hall conductance (1). This argument is very general, insensitive to the type of material, purity of sample, and strength of magnetic field etc. The crucial ingredients used in the argument are: gauge invariance, the existence of a gap, and the number of electrons have to be integral!

(d) We will not be able to study the fractional quantum hall effect in 8.06. However, I want to give you at least a some sense of it, at a qualitative level. At the very least, I want to convey how the discovery of this effect was even more of a surprise than the discovery of the integer quantum hall effect. The fractional quantum hall effect was discovered in 1982 by Tsui, Störmer and Gossard. They studied a very clean sample of the same sort in which the integer quantum hall effect had been discovered two years earlier. At a low temperature, and in a high enough magnetic field, they discovered a plateau in σ_H (1) with $\nu = \frac{1}{3}$ occurring over a range of filling fractions centered at

$$\frac{nhc}{eB} = \frac{1}{3} \ .$$

[3]Note that the above argument does not apply to degenerate states. For example, consider a free electron gas with $\nu = \frac{1}{3}$. From (b) one might have naively concluded that 1/3 of an electron is moved from one edge to the other. However in this case the conclusion does not apply since the states are highly degenerate: there are many different possibilities for 1/3 filling. By changing the flux from 0 to Φ_0, while the Hamiltonian is the same at the beginning and the end, electrons can move from occupied states to unoccupied states with essentially no cost of energy. The microscopic description of such changes is complicated since it depends on the initial state of the electrons and etc. The charge $Q = \frac{e}{3}$ should then in this case be understood as an average effect of a complicated bulk change.

If we attempt to describe this with 8.06 physics, we would say we have a one-third filled Landau level, and as such would have very many degenerate ground states to choose between, corresponding to the choice of which third of the states that make up the lowest Landau level we fill, and which two thirds we leave empty. This cannot describe the experimental result.

The correct description of the fractional quantum hall effect relies crucially on the Coulomb interaction between the electrons. The state described by the "Laughlin wave function", named after Robert Laughlin who proposed it as a way to understand the experimental results, cannot be described by first solving for single electron wave functions, and then filling some states while leaving others empty. In other words, it is not a direct product of single electron wave functions. *It is an intrinsic many-electron wave function.*[4]. However, we can understand at a qualitative level why its discovery required a very clean sample: in order for the fractional quantum hall effect to be observed, the effects of Coulomb repulsion between the electrons must dominate over the interaction between electrons and impurities. Since Tsui et al's discovery, plateaus have been seen at more and more fractional hall conductivities, for example at 1/3, 2/5, 3/7, 4/9, 5/11, 6/13, ... and 2/3, 3/5, 4/7, 5/9, 6/11, 7/13, ... and many more. It turns out that all these states are described by *non-degenerate* ground state wave functions, very different from the massively degenerate states we would construct à la 8.06, starting with single electron states, ignoring Coulomb interactions, and simply filling a fraction of a Landau level.

Applying the discussion of (a)–(c) to this case, we will now conclude that the system must have excitations with charge 1/3 ! This was a genuine revolutionary discovery. Rather than interpreting it as a 1/3 of an electron, this excitation has to do with the *collective* behavior of many electrons.

3. **Perturbation of the Three-Dimensional Harmonic Oscillator (18 points)**

The spectrum of the three-dimensional harmonic oscillator has a high degree of degeneracy. In this problem, we see how the addition of a perturbation to the Hamiltonian reduces the degeneracy. This problem is posed in such a way that you can work through it before we even begin to discuss degenerate perturbation theory in lecture.

Consider a quantum system described by the Hamiltonian

$$H = H_0 + H_1 \tag{2}$$

where

$$H_0 = \frac{1}{2m}\vec{p}^2 + \frac{1}{2}m\omega^2\vec{x}^2 \tag{3}$$

[4]Solving the Schrodinger equation with electron-electron interactions is certainly not possible. There are, say of order 10^{22}, electrons! Laughlin simply guessed the answer!

where $\vec{x} = (x_1, x_2, x_3)$ and $\vec{p} = (p_1, p_2, p_3)$. The perturbing Hamiltonian H_1 is given by

$$H_1 = KL_2 \tag{4}$$

where K is a constant and where $L_2 = x_3 p_1 - x_1 p_3$.

In parts (a)-(e) of this problem, we study the effects of this perturbation within the degenerate subspace of states which have energy $E = (5/2)\hbar\omega$ when $K = 0$.

(a) (3 points) Set $K = 0$. Thus, in this part of the problem $H = H_0$. Define creation and annihilation operators for "oscillator quanta" in the 1, 2 and 3 directions. Define number operators N_1, N_2, N_3. Denote eigenstates of these number operators by their eigenvalues, as $|n_1, n_2, n_3\rangle$. What is the energy of the state $|n_1, n_2, n_3\rangle$? How many linearly independent states are there with energy $E = (5/2)\hbar\omega$? [That is, what is the degeneracy of the degenerate subspace of states we are studying?]

(b) (3 points) Express the perturbing Hamiltonian H_1 in terms of creation and annihilation operators.

(c) (3 points) What is the matrix representation of H_1 in the degenerate subspace you described in part (a)?

(d) (4 points) What are the eigenvalues and eigenstates of H_1 in the degenerate subspace? What are the eigenvalues and eigenstates of $H = H_0 + H_1$ in the degenerate subspace?

(e) (3 points) What is the matrix representation of $H_0 + H_1$ in the degenerate subspace if you use the eigenvectors of H_1 as a new basis? (I.e. instead of the original $|n_1, n_2, n_3\rangle$ basis.)

[Note: As we shall see in part (f), this problem is "too simple" in important ways. The aspect of this problem which *will* generalize when we consider more generic perturbations is that if a perturbation breaks a degeneracy, then even an arbitrarily small but nonzero perturbation has qualitative consequences: it selects one particular choice of energy eigenvectors, within thepreviously degenerate subspace. In the present problem, this can be described as follows: if K were initially zero and you were happily using the $|n_1, n_2, n_3\rangle$ states as your basis of energy eigenstates, and then somebody "turns on" a very small but nonzero value of K, this forces you to make a qualitative change in your basis states. The "rotation" you must make from your previous energy eigenstates to the new states which are now the only possible choice of energy eigenstates is not a small one, even though K is arbitrarily small.]

(f) (2 points) $|\psi\rangle$ and $|\phi\rangle$ are eigenstates of H_0 with *different* energy eigenvalues. That is, $|\psi\rangle$ and $|\phi\rangle$ and belong to different degenerate subspaces. Show that $\langle\phi|H_1|\psi\rangle = 0$ for any two such states. Relate this fact to a statement you can make about the operators H_0 and H_1, without reference to states.

[The fact that $\langle\phi|H_1|\psi\rangle = 0$ if $|\psi\rangle$ and $|\phi\rangle$ and belong to different degenerate subspaces means that H_1 is a "non-generic" perturbation of H_0; a more general perturbation would not have this property. It is only for perturbations with this property that the analysis you have done above — which focusses on one degenerate subspace at a time — is complete. Notice also that in order to analyze $H = H_0 + H_1$, we did not have to assume that K was in any sense small. If H_1 were "generic", we would have had to assume that K was small in order to make progress.]

4. **A Delta-Function Interaction Between Two Bosons in an Infinite Square Well (8 points)**

   Do Griffiths problem 6.3.

   [The integrals that come up in this problem can certainly all be done by hand; however, there is also nothing wrong with doing them by Mathematica or the equivalent.]

5. **Anharmonic Oscillator (14 points)**

   Consider the anharmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} + \lambda x^3 \ ,$$

   treating the λx^3 term as a perturbation. [Hint: you should not find yourself doing any integrals as you do this problem; you should find yourself manipulating harmonic oscillator creation and annihilation operators and harmonic oscillator energy eigenstates.]

   (a) (3 points) Show that the first order shift in the ground state energy is zero. Calculate the shift to order λ^2.

   (b) (3 points) Calculate the ground state wave function to order λ. (You may just write your answer as a sum of harmonic oscillator states.)

   (c) (3 points) Sketch the potential $V(x)$ as a function of x for small λ. Is the state you found in (b) anything like the true ground state? What effect has perturbation theory failed to find?

   (d) (5 points) Now consider instead an anharmonic oscillator with

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} + \lambda x^4 \ ,$$

   treating the λx^4 term as a perturbation. Calculate the energy of the ground state to order λ. Sketch $V(x)$ for λ small and positive and for λ small and negative, and comment on what perturbation theory has told you in each case, and in each case comment on whether you think that perturbation theory has given a good approximation to the true ground state energy.