Quantum Physics III (8.06) Spring 2008
Assignment 8

April 8, 2008 **Due Friday April 18, 2008**

- Please remember to put your name **and section time** at the top of your paper.

Readings

- Griffiths Chapter 8 on the semiclassical approximation, and the Supplementary Notes on the Connection Formulae.
- Griffiths Chapter 10 on the adiabatic approximation.
- Note that I have not given you a problem on Berry's phase. The classic example is Griffith's Example 10.2, and I decided that it is too much to ask you to work through any example other than this one. This example will be done carefully in lecture. You should make sure you understand Berry's phase and this classic example, even though it will not appear on a problem set. If you want to do another example, try Griffiths' Problem 10.6.
- The article by Wick Haxton on the Solar Neutrino Problem that will soon be posted on the web page is *optional* extra reading.

Problem Set 8

1. **Tunnelling and the Stark Effect (18 points)**

   The Stark effect concerns the physics of an atom in an electric field. In this problem, we discuss the possibility that in an electric field, the electron in an atom can tunnel out of the atom, making the atomic bound states unstable. We consider this effect in a simpler one-dimensional analog problem.

   Supose an electron is trapped in a one-dimensional square well of depth V_0 and width d:

$$\begin{aligned} V(x) &= -V_0 \text{ for } |x| < d/2 \\ &= 0 \quad \text{for } |x| \geq d/2 \; . \end{aligned}$$

   Suppose a weak constant electric field in the x-direction with strength $\mathcal{E}$ is turned on. That is $V \to (V - e\mathcal{E}x)$. Assume throughout this problem that $e\mathcal{E}d \ll \hbar^2/2md^2 \ll V_0$.

(a) Set $\mathcal{E} = 0$ in this part of the problem. Estimate the ground state energy (ie the amount by which the ground state energy is above the bottom of the potential well) by pretending that the well is infinitely deep. (Because $\hbar^2/2md^2 \ll V_0$, this is a good approximation.) Use this estimate of the ground state energy in subsequent parts of the problem. Note that the true ground state energy is lower than what you've estimated, why?

(b) Sketch the potential with $\mathcal{E} \neq 0$ and explain why the ground state of the $\mathcal{E} = 0$ potential is no longer stable when $\mathcal{E} \neq 0$.

(c) Use the semiclassical approximation to calculate the barrier penetration factor for the ground state. [You should use the fact that $e\mathcal{E}d \ll \hbar^2/2md^2$ to simplify this part of the problem.]

(d) Use classical arguments to convert the barrier penetration factor into an estimate of the lifetime of the bound state.

(e) Now, lets put in numbers. Calculate the lifetime for $V_0 = 20$ eV, $d = 2 \times 10^{-8}$ cm and an electric field of 7×10^4 V/cm. Compare the lifetime you estimate to the age of the universe.

(f) Show that the lifetime goes like $\exp 1/\mathcal{E}$, and explain why this result means that this "instability" could not be obtained in any finite order of perturbation theory, treating $\mathcal{E}$ as a perturbation to the Hamiltonian.

2. **Quantum Mechanics of a Bouncing Ball (8 points)**

The semiclassical approximation can also be used to estimate the energy eigenvalues and eigenstates for potentials that cannot be treated exactly so easily. This problem is loosely based on Griffiths' 8.6. (See Griffiths' 8.5 if you'd like to learn how to treat this quantum mechanical problem exactly, using Airy functions.)

Consider the quantum mechanical analogue to the classical problem of a ball of mass m bouncing elastically on the floor, under the influence of a gravitational potential which gives it a constant acceleration g.

(a) Find the semiclassical approximation to the allowed energies E_n, in terms of m, g and $\hbar$.

(b) Estimate the zero point energy of a neutron "at rest" (ie in the quantum mechanical ground state) on a horizontal surface in the earth's gravitational field. Express your answer in eV. [This may sound artificial to you, but the experiment has been done. See V. V. Nesvizhevsky *et al.*, Nature **415**, 297 (2002) and arXiv:hep-ph/0306198 for an experimental measurement of the quantum mechanical ground state energy for neutrons bouncing on a horizontal surface in the earth's gravitational field. This experiment got a lot of press at the time, because it involves both gravity and quantum mechanics, which made for an eye catching press release. It of course has *nothing* to do with quantum gravity.]

(c) Now imagine dropping a ball of mass 1 gram from rest from a height of 1 meter, and letting it bounce. Do the 8.01 "calculation" of the classical energy of the ball. The quantum mechanical state corresponding to a ball following this classical trajectory must be a coherent superposition of energy eigenstates, with mean energy equal to the classical energy. How large is the mean value of the quantum number n in this state?

3. **Application of the Semiclassical Method to the Double Well Potential (20 points)**

Do Griffiths Problem 8.15.

This is not as difficult a problem as its length would indicate. Griffiths leads you through all the steps. This is an instructive problem in quantum dynamics. You should recall that this is the potential that we used to describe the physics of the ammonia molecule, early in 8.05. Back then, we had to wave our hands a little when we talked about tunnelling splitting the degeneracy between the even and odd states. Now, you can do this calculation for real.

Hint for (a) and (b): The steps suggested by Griffiths are: work out the wave function ψ_1 in region (i); from ψ_1 use the connection formulae at x_2 to obtain the wave function ψ_2 in regions (ii); use ψ_2 and the connection formulae at x_1 to obtain the wave function ψ_3 in region (iii). (8.59) can be found by requiring that ψ_3 should satisfy $\psi_3(0) = 0$ or $\psi_3'(0) = 0$ at $x = 0$.

It is a bit easier (and more transparent) to use a slightly different approach from what Griffiths suggests. Given that the wave function should be an even or odd function of x, the wave function in region (iii) can be written down immediately. For example in the even case,

$$\psi(x) = \frac{C}{\sqrt{\kappa(x)}} \cosh\left[\frac{1}{\hbar}\int_0^x dy\, \kappa(y)\right], \qquad -x_1 < x < x_1 \tag{1}$$

using our standard notations. (**??**) is an example where by symmetry, the exponentially small piece in a classically forbidden region is known exactly. The wave function ψ_2 in region (ii) then can be obtained using two ways: from ψ_1 in region (i) via connection formulae at x_2, or from ψ_3 in region (iii) via connection formulae at x_1. The consistency of two wave functions leads to equation (8.59) of Griffiths.

4. **Adiabatic Spin Rotation (5 points)**

Consider a spin one-half particle at rest, with its spin free to rotate in response to a time-dependent magnetic field. The Hamiltonian of the system is

$$H = -\frac{2\mu_0}{\hbar}\vec{S}\cdot\vec{B}(t) \ .$$

At $t = 0$ there is a magnetic field $\vec{B}(0) = (0, 0, B_0)$, and the spin is aligned along the magnetic field, with

$$|\psi(0)\rangle = |\uparrow\rangle \ .$$

Now suppose that the magnetic field is very slowly decreased to zero and then increased in the opposite direction,

$$\vec{B}(t) = (0, 0, B_0 - \beta t) \ ,$$

until at time $t_f = 2B_0/\beta$, we have $\vec{B}(t_f) = (0, 0, -B_0)$. In addition, assume that there is a small, *constant* residual magnetic field in the xy-plane, $\delta\vec{B} = (B_x, B_y, 0)$. [After all, it would be unrealistic to assume that the magnetic field is *exactly* zero when $t = B_0/\beta$.]

Use the adiabatic theorem to show that the particle initially in the state $|\uparrow\rangle$ finishes in the state $|\downarrow\rangle$ with unit probability, regardless of the direction that $\delta\vec{B}$ points in the xy-plane. What is the condition on the parameters of the problem for the adiabatic theorem to apply?

[Aside: If the adiabatic theorem applies, the particle is in the state with spin parallel (as opposed to antiparallel) to the magnetic field at the end of the evolution just as at the beginning. This property is applied in the magnetic traps used by MIT atomic physicists to trap very cold gases of spin-polarized atoms. The magnetic field gradients are designed so that an atom which has its spin parallel to the local magnetic field $\vec{B}(x)$ experiences a force toward the center of the trap. Those atoms with spins antiparallel to $\vec{B}(x)$ feel a force which expels them from the trap. This raises a question: since $\vec{B}(x)$ varies in space. how do we ensure that as the atoms move within the trap their spins are always aligned with the local magnetic field? Answer: the atoms are moving slowly enough that the adiabatic theorem applies. You should now be able to explain the importance of the following design feature of the traps: since it is magnetic *gradients* which exert the trapping forces, you might think that there would be no problem if at one point in the trap, $\vec{B} = 0$; in fact, this *does* cause problems, and the traps are designed to have $\vec{B} \neq 0$ everywhere. Explain.]

5. **Engineering Adiabatic Transitions (9 points)**

This problem is similar to the last one, except that now the particle has spin one, and therefore has three eigenstates $|+\rangle$, $|0\rangle$ and $|-\rangle$.

The Hamiltonian is given by

$$H = -\frac{2\mu_0}{\hbar}\vec{S}\cdot\vec{B}(t) - \frac{c}{\hbar^2}S_z^2 \ .$$

The last term represents some small term in the environment of the spin which favors the states $|\pm\rangle$ over the state $|0\rangle$.

(a) Write the operators S_x, S_y and S_z for this problem. (You should be able to find them in your 8.05 notes or in one of your books.)

(b) Suppose that the magnetic field is as in the previous problem, with a small constant field in the x-direction and a large time-dependent field in the z-direction:

$$\vec{B}(t) = (B_x, 0, B_0 - \beta t) \ .$$

Assume the following hierarchy of energies:

$$\mu_0 B_0 \gg c \gg \mu_0 B_x \gg \hbar\beta/B_x \ .$$

Sketch the energy levels of this system as a function of time.

(c) Suppose the system starts out in the state $|-\rangle$ at $t = 0$. Show that upon assuming the hierarchy of energies above, this state evolves to a state that is approximately equal to $|0\rangle$ and then evolves to $|+\rangle$. Explain how you would alter the time dependence of $\vec{B}(t)$ in order that the initial state transforms into $|0\rangle$ at late times.