Quantum Physics III (8.06) Spring 2008
Solution Set 4
March 4, 2008

1. Landau levels without choosing a gauge (5 points)

(a) (3 points) The Hamiltonian is given by:

\[
H = \frac{1}{2}m(v_x^2 + v_y^2) \tag{1}
\]

\[
= \frac{p^2}{2m} + \frac{1}{2}m\frac{1}{c^2}Q^2 \tag{2}
\]

which has the form of the Hamiltonian for a harmonic oscillator. We will choose \(d = m\) so that the harmonic oscillator has mass \(m\). Imposing the canonical commutation relation \([Q, P] = i\hbar\) then fixes \(c\) to be \(c = mc/qB = 1/\omega_L\). We thus find that it is harmonic oscillator with frequency \(\omega_L\).

(b) (2 points) The annihilation and creation operators are given by:

\[
a = \sqrt{\frac{m}{2\hbar \omega_L}}(v_x + iv_y) \tag{3}
\]

\[
a^\dagger = \sqrt{\frac{m}{2\hbar \omega_L}}(v_x - iv_y) \tag{4}
\]

which correctly satisfy \([a, a^\dagger] = 1\).

2. Landau levels in symmetric gauge (18 points)

(a) (4 points) Since \(\vec{p} = -i\hbar \frac{\partial}{\partial \vec{x}}\) we can write the the complex derivatives as

\[
\partial_z = \frac{1}{2}(\partial_x - i\partial_y) = \frac{i}{2\hbar}(p_x - ip_y)
\]

\[
\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y) = \frac{i}{2\hbar}(p_x + ip_y) \tag{5}
\]

Also we observe that

\[
\frac{A_y - iA_x}{Bl_0} = \frac{x}{2l_0} + i\frac{y}{2l_0} = \frac{z}{2l_0} \tag{6}
\]

Plugging (5), (6) into the definition of \(a\) and using \(v_x = \frac{1}{m}(p_x - qA_x/c)\), we find

\[
a = \sqrt{\frac{m}{2\hbar \omega_L}}(v_x + iv_y)
\]

\[
= \sqrt{\frac{m}{2\hbar \omega_L}} \frac{1}{m} \left( -\frac{q}{c}(A_x + iA_y) + (p_x + ip_y) \right)
\]

\[= l_0 \frac{q}{\sqrt{2}} \left( -\frac{A_x - iA_y}{Bl_0^2} + \frac{1}{\hbar}(p_x + ip_y) \right)\]

\[= -\frac{i}{\sqrt{2}} \left( \frac{z}{2l_0} + 2l_0 \partial_z \right)\]
(b) (3 points) Plugging the definition of $a$ in the annihilation condition $a|\psi\rangle = 0$ we get

$$\partial_z \psi = -\frac{z}{4l_0^2} \psi.$$ 

This can be viewed as an ordinary differential equation for $\psi$ with $\bar{z}$ as independent variable and $z$ as a parameter. The general solution is

$$\psi(z, \bar{z}) = f(z)e^{-\frac{z\bar{z}}{4l_0^2}}$$ 

where $f(z)$ is an arbitrary function of $z$.

(c) (6 points) The basis of the ground state wave functions is

$$\psi_n = N_n z^n e^{-\frac{z\bar{z}}{4l_0^2}}$$

We compute the normalization of $\psi_n$ by first going to the polar coordinates $r, \phi$. Then the integral over $\phi$ is easily computed. To take the integral over $r$ we perform another change of variables $\rho = r^2$:

\begin{align*}
1 &= \langle \psi_n | \psi_n \rangle \\
&= \frac{N_n^2}{2} \int dz d\bar{z} (z\bar{z})^n e^{-\frac{z\bar{z}}{4l_0^2}} \\
&= N_n^2 \int_0^\infty dr \int_0^{2\pi} d\phi r^{2n+1} e^{-\frac{r^2}{4l_0^2}} \\
&= \pi N_n^2 \int_0^\infty d\rho \rho^n e^{-\frac{\rho}{4l_0^2}} \\
&= \pi N_n^2 (-1)^n \left. \frac{\partial^n}{\partial \alpha^n} \left( \int_0^\infty e^{-\alpha \rho} d\rho \right) \right|_{\alpha = \frac{1}{4l_0^2}} \\
&= \pi N_n^2 (-1)^n \left. \frac{\partial^n}{\partial \alpha^n} \left( \frac{1}{\alpha} \right) \right|_{\alpha = \frac{1}{4l_0^2}} \\
&= \pi N_n^2 (-1)^n \left. \left( -1 \right)^n \frac{n!}{\alpha^{n+1}} \right|_{\alpha = \frac{1}{4l_0^2}} \\
&= \pi N_n^2 n! (2l_0^2)^{n+1}
\end{align*}

and therefore

$$N_n = (\pi n!2^{n+1}l_0^{2n+2})^{-\frac{1}{2}}.$$ (7)

Note: In the above calculation, we could also have used the fact that $\Gamma(x) = \int_0^\infty dt t^{-1} e^{-t}$.

This gives

$$\int_0^\infty d\rho \rho^n e^{-\frac{\rho}{4l_0^2}} = (2l_0^2)^{n+1} \Gamma(n+1) = (2l_0^2)^{n+1} n!.$$ 

The contour plots of $|\psi_n(x, y)|$ for $n = 0, 4, 10, 25$ are shown on the figures 1, 2, 3 and 4 respectively with comments in the captions.
Figure 1: $|\psi_0(x, y)|$ is simply a gaussian which has a peak at 0.
Figure 2: We see, that for $n = 4$ the probability to detect the particle is concentrated in the circular area around the origin.
Figure 3: The case of $n = 10$ is qualitatively similar to the $n = 4$ case, the radius of the ring grows roughly as the square root of $n$. 
Figure 4: The case of $n = 25$ is qualitatively similar to the $n = 10$ case.
(d) (5 points) The angular momentum \( L \) in complex notations is
\[
L_z = xp_y - yp_x
= (-i\hbar)\frac{1}{2} ((z + \bar{z})(\partial_z - \partial_{\bar{z}}) + (z - \bar{z})(\partial_z + \partial_{\bar{z}}))
= \hbar (z\partial_z - \bar{z}\partial_{\bar{z}}).
\]

Let us compute the commutator of angular momentum \( L_z \) with raising operator \( a^\dagger \)
\[
[L_z, a] = -\frac{i\hbar}{\sqrt{2}} \left[ z\partial_z - \bar{z}\partial_{\bar{z}}, \frac{z}{2l_0} + 2l_0\partial_z \right]
= -\frac{i\hbar}{\sqrt{2}} \left( \left[ z\partial_z, \frac{z}{2l_0} \right] - [z\partial_z, 2l_0\partial_z] \right)
= -\frac{i\hbar}{\sqrt{2}} \left( \frac{z}{2l_0} + 2l_0\partial_z \right)
= \hbar a^\dagger.
\]

Taking the Hermitian conjugate:
\[
[L_z, a^\dagger] = -\hbar a.
\]

The commutator of angular momentum with the Hamiltonian is
\[
[L_z, H] = [L_z, \hbar\omega_L(a^\dagger a + \frac{1}{2})]
= \hbar\omega_L \left( [L_z, a^\dagger]a + a^\dagger[L_z, a] \right)
= 0.
\]

Acting with \( L_z \) on \( \psi_n \) we obtain
\[
L_z \psi_n = \hbar N_n (z\partial_z - \bar{z}\partial_{\bar{z}}) e^{-\frac{x^2 + y^2}{2l_0^2}}
= \hbar N_n z e^{-\frac{r^2}{2l_0^2}}
= \hbar n \psi_n.
\]

## 3. Counting States in the lowest Landau Level in the symmetric gauge (7 points)

(a) (2 points) We wish to find \( P_n(r) \) such that
\[
\int_0^\infty P_n(r)dr = 1. \tag{8}
\]

First note that \( |\psi_n|^2 = N_n^2 (x^2 + y^2)^n e^{-\frac{x^2 + y^2}{2l_0^2}} = N_n^2 r^{2n} e^{-\frac{r^2}{2l_0^2}} \), which is independent of \( \phi \). Now from the normalization of \( \psi_n \), we have
\[
1 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy |\psi_n|^2
= \int_0^\infty rdr \int_0^{2\pi} d\phi |\psi_n(r)|^2
= \int_0^\infty dr \left( 2\pi r |\psi_n(r)|^2 \right). \tag{9}
\]
Here in the second equality, we have converted from cartesian to polar coordinates and $N_n$ is given by equation (7). The factor of $r$ arises in the usual way from the jacobian of the transformation. In the third equality, we do the integral over $\phi$ as the integrand is independent of it. Comparing (8) and (9), we can immediately read of

$$P_n(r) = 2\pi r |\psi_n(r)|^2.$$  \hfill (10)

(b) (3 points) The probability density is

$$P_n(r) = 2\pi r |\psi_n^2| = 2\pi N_n^{2n+1} r^2 e^{-\frac{r^2}{2l_0^2}}$$

Taking derivative with respect to $r$ and setting it to zero we get

$$(2n + 1) - \frac{r_{\text{max}}^2}{l_0^2} = 0.$$  

Therefore

$$r_{\text{max}} = l_0\sqrt{2n + 1}$$

(c) (2 points) The largest value of $n_{\text{max}}$ allowed by the size of the material. This can be determined from

$$l_0\sqrt{2n_{\text{max}} + 1} \approx R.$$  

Therefore

$$n_{\text{max}} \approx \frac{1}{2} \frac{R^2}{l_0^2} = \frac{\pi R^2 qB}{\hbar c} = \frac{\Phi}{\Phi_0}.$$  

4. Coherent state in the symmetric gauge (12 points)

This problem is wrong. The following was what I intended it to be done. But the method is flawed. $\psi(t)$ described below does NOT correspond to the time evolution of the state in part (a). The reason is that there are an infinite number of solutions to the equation (22) below. Suppose $\psi_1(z, \bar{z})$ is a solution to the equation, then $f(z)\psi_1(z, \bar{z})$ is also a solution for any holomorphic function $f(z)$ (which depends on $z$ only). I had thought one was able to determine $f(z)$ using the initial condition. But I was wrong. From (22) below one can NOT the conclusion that (23) is the time evolution of the wave function in (a). Also a quick way to see that the state (23) cannot be the right coherent state describing classical circular motion is to check that the expectation values of the velocity operators in it do not correspond to those of a classical particle undergoing circular motion.

I put on the 8.06 website a separate note describing a proper way of finding the coherent states corresponding to the classical circular motion.

(a) (2 points) $\Psi_0$ is given by:

$$\Psi_0(z, \bar{z}) = \frac{1}{l_0\sqrt{2\pi}} e^{-\frac{(z-z_0)^2+(\bar{z}-\bar{z}_0)^2}{2l_0^2}}.$$  \hfill (11)
(b) (10 points) We follow the steps suggested in the Problem Set:

i. Using the operator $a$ given by expression (9) in the Problem Set, we can show:

$$a\Psi_0 = -\frac{i}{\sqrt{2}} (\frac{z}{2l_0} + 2l_0 \partial_z) \frac{1}{l_0 \sqrt{2\pi}} e^{-\frac{(z - z_0)(\bar{z} - \bar{z}_0)}{4l_0^2}}$$

$$= -i \frac{z_0}{2l_0 \sqrt{2}} \Psi_0$$

$$\implies \Psi_0 \text{ is an eigenstate of } a \text{ with eigenvalue } -i \frac{z_0}{2l_0 \sqrt{2}}.$$ 

ii. Consider the operator $a$ in the Heisenberg picture:

$$a_H(t) = e^{iHt/\hbar} a e^{-iHt/\hbar}.$$  

Its time evolution is governed by:

$$\frac{da_H}{dt} = \frac{1}{i\hbar}[a_H, H], \quad a_H(0) = a,$$

which can be solved to give us

$$a_H(t) = e^{-i\omegaLt} a.$$ 

Since the Hamiltonian is time-independent, the wave function at time $t$ is given by

$$\psi(t) = e^{-iHt/\hbar} \Psi_0.$$ 

Then,

$$a\psi(t) = ae^{-iHt/\hbar} \Psi_0$$

$$= e^{-iHt/\hbar} e^{iHt/\hbar} ae^{-iHt/\hbar} \Psi_0$$

$$= e^{-iHt/\hbar} a_H(t) \Psi_0$$

$$= e^{-iHt/\hbar} e^{-i\omegaLt} a \Psi_0$$

$$= e^{-iHt/\hbar} e^{-i\omegaLt} \left( -i \frac{z_0}{2\sqrt{2l_0}} \Psi_0 \right)$$

$$= -i \frac{z_0 e^{-i\omegaLt}}{2\sqrt{2l_0}} \psi(t).$$

In the third equality we have used $a_H(t) = e^{iHt/\hbar} a e^{-iHt/\hbar}$ and in the fourth equality, we used (16). Hence, $\psi(t)$ is an eigenstate of $a$ with eigenvalue $-i \frac{z_0 e^{-i\omegaLt}}{2\sqrt{2l_0}}$.

iii. Therefore, $\psi(t)$ satisfies exactly the same equation as $\Psi_0$ with $z_0$ replaced by $z_0(t) \to z_0 e^{-i\omegaLt}$. Thus

$$\psi(t) = \exp \left( -\frac{(z - z_0(t))(\bar{z} - \bar{z}_0(t))}{4l_0^2} \right).$$

5. Translation Invariance in a Uniform Magnetic Field (12 points)

(a) (2 points) Since $\vec{A} = (-By, 0, 0)$, translations in the $y$ direction: $y \to y + b$ (where $b$ is a constant), imply:

$$\vec{A} \to \vec{A} + (-Bb, 0, 0).$$
6. Off-diagonal conductance in two dimensions (6 points)

(a) (3 points) Using the usual formula for the inverse of a $2 \times 2$ matrix, we write

$$
\sigma = \frac{1}{\rho_0^2 + \rho_H^2} \begin{pmatrix} \rho_0 & \rho_H \\ -\rho_H & \rho_0 \end{pmatrix},
$$

(b) (3 points) If $\psi(x, y)$ is a solution to the Schrodinger equation with gauge $\vec{A} = (-By, 0, 0)$, then $\psi(x, y + b)$ is a solution to the Schrodinger equation with gauge $\vec{A}' = (-B(y + b), 0, 0)$. Then, we can gauge transform $\vec{A}'$ back to $\vec{A} = (-By, 0, 0)$ using $f$ from part (a). The wave function after the gauge transformation is given by:

$$
\tilde{\psi}(x, y) = \psi(x, y + b)e^{-iBx/B},
$$

which satisfies the original Schrodinger equation with $\vec{A} = (-By, 0, 0)$.

(c) (3 points)

$$
V_b \psi(x, y) = \psi(x, y + b)e^{iBx/B},
$$

where we have used the translation operator to produce translations in the $y$ direction and in the last line we have used $[x, p_y] = 0$. Similarly, $U_a = e^{i\pi Bx/a}$.

(d) (4 points)

$$
[U_a, V_b] = [e^{i\pi Bx/a}, e^{iBx/B}],
$$

where to go from the first to the second line we have used $[AB, C] = [A[B, C] + [A, C]B$ and from the second to the third line we have used $[p_x, p_y] = 0$. Using $e^{A+B} = e^{A+e^{i\pi B/a}[A,B]}$ (valid when $[A, B]$ constant) and $[x, p_x] = i\hbar$ we find:

$$
[e^{i\pi Bx/a}, e^{iBx/B}] = \frac{e^{i\pi Bx/a} + e^{iBx/B}}{-i\hbar/a} - e^{i\pi Bx/a} + \frac{i\pi Bx/a}{\hbar} e^{iBx/B} e^{i\pi Bx/a}[x, p_x],
$$

$$
\frac{e^{i\pi Bx/a} + e^{iBx/B}}{-i\hbar/a} - e^{i\pi Bx/a} + \frac{i\pi Bx/a}{\hbar} e^{iBx/B} e^{i\pi Bx/a}[x, p_x] - 2ie^{i\pi Bx/a} e^{iBx/B} \sin \left( \frac{\pi Bx/a}{\hbar} \right) = -2ie^{i\pi Bx/a} e^{iBx/B} \sin \left( \frac{\pi Bx/a}{\hbar} \right) .
$$

So, we see that when $ab = n\hbar/a$, ie. when flux = $abB = nhc/q$, where $n$ is an integer, $\sin \left( \frac{\pi Bx/a}{\hbar} \right) = 0$ and hence $[U_a, V_b] = 0$.

6. Off-diagonal conductance in two dimensions (6 points)

(a) (3 points) Using the usual formula for the inverse of a $2 \times 2$ matrix, we write
which gives us
\[ \sigma_0 = \frac{\rho_0}{\rho_0^2 + \rho_H^2} \quad \text{and} \quad \sigma_H = \frac{\rho_H}{\rho_0^2 + \rho_H^2}. \]

(b) \textbf{(2 points)} If \( B = 0 \), then \( \rho_H = 0 \) and \( \sigma_0 = 1/\rho_0 \). Hence as \( \rho_0 \rightarrow 0 \), \( \sigma_0 \rightarrow \infty \).

If \( \rho_H \) is non vanishing, then
\[
\lim_{\rho_0 \to 0} \sigma_0 = \lim_{\rho_0 \to 0} \frac{\rho_0}{\rho_0^2 + \rho_H^2} = 0.
\]

Thus we can have both \( \sigma_0 \) and \( \rho_0 \) equal to 0 in the presence of a non vanishing \( B \) field.

(c) \textbf{(1 point)} From equation (20) of the psset [resistivity] = [Electric field]/[current density] = [voltage]/[current] = [resistance], because [electric field] = [voltage]/[length] and [current density] = [current]/[length]. Hence, the resistance and resistivity have the same dimensions. This is not true in three dimensions because current density there is current per unit area while the dimensions of voltage remain the same.