Quantum Physics III (8.06) Spring 2008
Solution Set 9

May 2, 2008

1. **Brick in a Square Well (8 points)**

   The ground state and first excited state wavefunctions of the unperturbed system are:

$$\psi_0 = \sqrt{\frac{2}{a}}\sin(\pi x/a) \qquad (1)$$

$$\psi_1 = \sqrt{\frac{2}{a}}\sin(2\pi x/a), \qquad (2)$$

   respectively. The relevant off-diagonal matrix element is $H'_{21} = \int_0^{a/2} dx\, \frac{2V_0}{a}\sin(\pi x/a)\sin(2\pi x/a) = \frac{4V_0}{3\pi}$. Note that in this case the diagonal matrix elements are also non-zero: $H'_{11} = H'_{22} = V_0/2$. Hence the appropriate equations to use are [9.19], [9.20], [9.21] of Griffiths 2nd edition, instead of [9.13]. The phase ϕ is zero since H'_{11} and H'_{22} are equal. Therefore d_1 and d_2 obey exactly the same equations as c_1 and c_2 would obey if we had zero diagonal matrix elements (equation [9.13]).

   So $d_2(T) = -\frac{i}{\hbar}\int_0^T dt\, H'_{21}e^{i\omega_{21}t}$, where $\omega_{21} = \frac{1}{\hbar}(E_2 - E_1) = \frac{3\hbar\pi^2}{2ma^2}$. Doing the integral, we find

$$d_2(T) = -\frac{8ma^2V_0}{9\pi^3\hbar^2}e^{i\left(\frac{3\hbar\pi^2}{4ma^2}\right)T}2i\sin\left(\frac{3\hbar\pi^2}{4ma^2}T\right).$$

   Since d_2 and c_2 only differ by a phase which does not contribute to the probabilities, the probability of measuring the energy to be E_2 is

$$P = |d_2|^2 = \left(\frac{16ma^2V_0}{9\pi^3\hbar^2}\right)^2\sin^2\left(\frac{3\hbar\pi^2}{4ma^2}T\right).$$

2. **A Time-Dependent Two State System (14 points)**

   (a) (4 points) Perturbation theory gives

$$c_1 = \frac{1}{i\hbar}\int_{-\infty}^{\infty} dt\, v(t)e^{2iEt/\hbar},$$

   and therefore the transition probability is (to lowest order)

$$P = \frac{1}{\hbar^2}\left|\int_{-\infty}^{\infty} dt\, v(t)e^{2iEt/\hbar}\right|^2.$$

   (b) (10 points) When $E = 0$, the eigenstates of the Hamiltonian are $|+\rangle \equiv \frac{1}{\sqrt{2}}[|1\rangle + |2\rangle]$ and $|-\rangle \equiv \frac{1}{\sqrt{2}}[|1\rangle - |2\rangle]$. To evolve them in time, we have

$$|+\rangle_t = e^{-\frac{i}{\hbar}\int_{t'}^t dt\, v(t)}|+\rangle_{t'}, \qquad |-\rangle_t = e^{\frac{i}{\hbar}\int_{t'}^t dt\, v(t)}|-\rangle_{t'}.$$

Since $|1\rangle_t = \frac{1}{\sqrt{2}}[|+\rangle_t + |-\rangle_t]$, and similarly for $|2\rangle_t$, we find that the overlap ${}_{\infty}\langle 1|2\rangle_{-\infty}$ is

$$\begin{aligned} {}_{\infty}\langle 1|2\rangle_{-\infty} &= \frac{1}{2}\left[e^{\frac{i}{\hbar}\int_{-\infty}^{\infty} dt\, v(t)} - e^{\frac{-i}{\hbar}\int_{-\infty}^{\infty} dt\, v(t)}\right] \\ &= i\sin\left(\frac{1}{\hbar}\int_{-\infty}^{\infty} dt\, v(t)\right). \end{aligned}$$

The exact transition probability is thus

$$P = \sin^2\left(\frac{1}{\hbar}\int_{-\infty}^{\infty} dt\, v(t)\right).$$

The perturbative calculation of part (a) yields in this case

$$P = \frac{1}{\hbar^2}\left(\int_{-\infty}^{\infty} dt\, v(t)\right)^2.$$

This is the first term in a small-v expansion of the exact result above. (Actually, strictly speaking, we need not only v but also $\int dt\, v$ to be small.)

3. **Excitation of a Hydrogen atom (10 points)**

We choose the z-axis along $\vec{E}_0$, so that the perturbing Hamiltonian is $H' = -eE_0 z e^{-\gamma t}$. We need the matrix elements $\langle 100|H'|21m\rangle$. But we know (from previous problem sets) that the matrix element $\langle 100|z|21m\rangle = 0$ vanishes for all m besides $m = 0$, so we only need to calculate $\langle 100|H'|210\rangle$. Letting $\zeta = \frac{2^7\sqrt{2}}{3^5}$ (you do not need to compute this number), this matrix element works out to be

$$H'_{1s;2p0} = -\zeta a_0 e E_0 e^{-\gamma t}.$$

Then, to first order,

$$c_{2p0} = \frac{1}{i\hbar}\int_0^t dt\, (-\zeta a_0 e E_0) e^{-(\gamma - i\omega_{21})t},$$

with $\hbar\omega_{21} = E_2 - E_1 = \frac{3\hbar^2}{8m_e a_0^2}$. Performing the integral, we have

$$\begin{aligned} c_{2p0}(t) &= \frac{\zeta i a_0 e E_0}{\hbar(i\omega_{21} - \gamma)}\left(e^{-(\gamma - i\omega_{21})t} - 1\right) \\ &\to \frac{\zeta a_0 e E_0}{\hbar}\,\frac{i\gamma - \omega_{21}}{\gamma^2 + \omega_{21}^2} \qquad \text{as } t \to \infty. \end{aligned}$$

The transition probability is thus

$$|c_{2p0}|^2 = \frac{(\zeta a_0 e E_0)^2}{\hbar^2(\gamma^2 + \omega_{21}^2)}.$$

4. **Decay of the Three Dimensional Harmonic Oscillator (14 points)**

(a)(2 points) In CGS units, the transition rate due to spontaneous emission is given by

$$A = \frac{4\omega^3 |\langle 1s|q\vec{r}|2p\rangle|^2}{3\hbar c^3}.$$

Here, of course, the frequency of the harmonic oscillator ω is exactly the same as the frequency determined by the energy difference between the two states, ω_{ba}.

(b) (8 points) The eigenstates of definite angular momentum L_z with one excitation are

$$\begin{aligned}\phi_{m=0} &= |001\rangle \\ \phi_{m=1} &= \frac{1}{\sqrt{2}}\left(|100\rangle + i|010\rangle\right) \\ \phi_{m=-1} &= \frac{1}{\sqrt{2}}\left(|100\rangle - i|010\rangle\right),\end{aligned}$$

while $\phi_s = |000\rangle$. It is now straightforward to see that $|\langle 1s|q\vec{r}|2p\rangle|^2$ is independent of m_z: since $x_i = \sqrt{\frac{\hbar}{2m\omega}}(a_i + a_i^\dagger)$, we find

$$\begin{aligned}\langle 000|\vec{r}|001\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\hat{z}, \\ \langle 000|\vec{r}\left(|100\rangle + i|010\rangle\right) &= \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{x} + i\hat{y}\right), \\ \langle 000|\vec{r}\left(|100\rangle - i|010\rangle\right) &= \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{x} - i\hat{y}\right),\end{aligned}$$

and therefore

$$|\langle 1s|q\vec{r}|2p\rangle|^2 = \frac{q^2\hbar}{2m\omega},$$

independent of m_z. (We could have proved this in a more abstract fashion by appealing to the isotropy of the potential, but it is instructive to work it out explicitly as we have here.)

(c) (2 points) Plugging the result of part (b) into that of part (a), we find

$$A = \frac{2q^2\omega^2}{3mc^3}.$$

(d) (2 points) The lifetime is given by the inverse of the transition rate, so

$$\tau_{2p} = \frac{1}{A} = \frac{3mc^3}{2q^2\omega^2}.$$

5. **A wavefront crossing a bound particle (14 points)**

(a) (4 points) To first order,

$$\begin{aligned}c_f &= \frac{1}{i\hbar}\int_{-\infty}^{\infty} dt\, e^{i\omega_{fi}t}\int_{-\infty}^{\infty} dx\, a\delta(x - ct)u_i^*(x)u_f(x) \\ &= \frac{a}{i\hbar}\int_{-\infty}^{\infty} dt\, e^{i\omega_{fi}t}u_i^*(ct)u_f(ct).\end{aligned}$$

The transition probability is then

$$P_f = |c_f|^2 = \left(\frac{a}{\hbar c}\right)^2 \left|\int_{-\infty}^{\infty} dy\, e^{i\omega_{fi}y/c}u_i^*(y)u_f(y)\right|^2.$$

(b) (5 points) Setting $V_\omega(x) = \frac{a}{c} e^{i\omega x/c}$, we can write

$$a\delta(x - ct) = \int \frac{d\omega}{2\pi} V_\omega(x) e^{-i\omega t}.$$

With this expansion for V, we find

$$\begin{aligned} c_{f,\omega} &= \frac{1}{i\hbar} \int \frac{dt}{2\pi}\, e^{i(\omega_{fi}-\omega)t} \int dx\, V_\omega(x) u_i^*(x) u_f(x) \\ &= \frac{1}{i\hbar} \delta(\omega - \omega_{fi}) \langle f|V_\omega|i\rangle. \end{aligned}$$

Now, $c_f = \int d\omega c_{f,\omega}$, and the integral is trivial because of the δ-function, so we find

$$c_f = \frac{a}{i\hbar c} \langle f| e^{i\omega_{fi} x/c} |i\rangle,$$

which is indeed the same result that we got in part (a).

The moral of the story is that we can reduce (almost) every problem, using Fourier transforms, to a sum of harmonic oscillator problems that we know how to do.

(c) (5 points) Using the square well wave functions, the matrix element $\langle f| e^{i\omega_{fi} x/c} |i\rangle$ becomes

$$\frac{2}{d} \int_0^d dy\, e^{i\omega_{fi} y/c} \sin\left(\frac{\pi y}{d}\right) \sin\left(\frac{2\pi y}{d}\right) = -\frac{16 i\beta(1 + e^{2\pi i\beta})}{(9 - 40\beta^2 + 16\beta^4)\pi}.$$

thus, the transition probability is

$$P = \frac{4\alpha^2}{\pi^2} \frac{\beta^2 \cos^2 \pi\beta}{(\frac{9}{16} - \frac{5}{2}\beta^2 + \beta^4)^2}.$$

We maximize the above to find that P_{max} occurs at $\beta = 0.97$. (I used Mathematica.) This makes physical sense: the natural timescale of the system is $\frac{2\pi}{\omega}$, while the time it takes light to cross the system is $\tau = \frac{d}{c} = \frac{2\pi\beta}{\omega}$. So, when $\beta \simeq 1$, the two time scales are approximately equal, there is maximal overlap between the pulse and the system, and the probability of making a transition is amplified.