

Massachusetts Institute of Technology
Physics Department

Physics 8.20
Special Relativity
2:30 pm

IAP 2008
February 1, 2008

FINAL EXAM

Instructions

- You have 2.5 hours for this test. The exam finishes at 5:00pm.
- There are five problems on this test, worth a total of 100 points.
- Calculators and one page of notes are allowed, but no other materials.
- Please do ALL your work in the white book that has been provided, remembering to write your name on the front of the white booklet.
- If you miss part (a) of a problem and need the answer to (a) to work part (b), you should feel free to express your answer to (b) in terms of the answer you should have obtained for part (a).
- If you have any questions please feel free to ask the proctor.

Good luck!

Information

Useful approximations:

$$c \approx 3 \times 10^8 \text{ m/s}$$

$$1 \text{ year} \approx \pi \times 10^7 \text{ sec.}$$

Boost (along the x -axis) and its inverse. Suppose $V = (v_0, v_x, v_y, v_z)$ is a four vector. Then

$$\begin{aligned} v'_x &= \gamma(v_x - \beta v_0) & v_x &= \gamma(v'_x + \beta v'_0) \\ v'_y &= v_y & v_y &= v'_y \\ v'_z &= v_z & v_z &= v'_z \\ v'_0 &= \gamma(v_0 - \beta v_x) & v_0 &= \gamma(v'_0 + \beta v'_x) \end{aligned}$$

where $\beta = v/c$, and $\gamma = 1/\sqrt{1 - \beta^2}$.

Velocity addition (relative motion along the x -axis):

$$\begin{aligned} u'_x &= \frac{u_x - v}{1 - u_x v/c^2} \\ u'_y &= \frac{u_y}{\gamma(1 - u_x v/c^2)} \\ u'_z &= \frac{u_z}{\gamma(1 - u_x v/c^2)} \end{aligned}$$

Lorentz transformation of acceleration:

The α_x and α_y be the components of the acceleration of an object in its instantaneous rest frame, Σ' , which has velocity v in the x -direction as observed in Σ .

$$\begin{aligned} a_x &= \frac{dv_x}{dt} = \frac{1}{\gamma^3(v)} \alpha_x \\ a_y &= \frac{dv_y}{dt} = \frac{1}{\gamma^2(v)} \alpha_y \end{aligned}$$

Uniform acceleration:

t , x , β , and γ refer to the lab frame; τ is the proper time of the accelerated observer. g is the proper acceleration.

$$\begin{aligned} \beta(t) &= \frac{gt/c}{\sqrt{1 + (gt/c)^2}} \\ x^2(t) - c^2 t^2 &= \frac{c^4}{g^2} \quad \text{or} \quad x = \frac{c^2}{g} \sqrt{1 + (gt/c)^2} \\ t &= \frac{c}{g} \sinh(g\tau/c) \end{aligned}$$

Doppler shift:

Longitudinal

$$\nu = \sqrt{\frac{1 + \beta}{1 - \beta}} \nu_0$$

Transverse

$$\nu = \sqrt{1 - \beta^2} \nu_0$$

Relativistic energy and momentum:

$$\vec{p} = m\gamma\vec{v} \quad E = m\gamma c^2 \quad E^2 = p^2 c^2 + m^2 c^4 \quad \text{K.E.} = E - mc^2$$

Quadratic equation:

$$ax^2 + bx + c = 0$$

$$x = \frac{1}{2a} \left(-b \pm \sqrt{b^2 - 4ac} \right)$$

Binomial expansion:

$$(1 + a)^b = 1 + ba + \frac{b(b-1)}{2} a^2 + \dots$$

Approximations:

For $\epsilon \ll 1$,

$$\frac{1}{1 - \epsilon} \approx 1 + \epsilon \quad \sqrt{1 - \epsilon} \approx 1 - \frac{\epsilon}{2}$$

Hyperbolic functions:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Identities:

$$\cosh^2 x - \sinh^2 x = 1, \quad 1 - \tanh^2 x = \frac{1}{\cosh^2 x}$$

Some integrals:

$$\int dx \sin x = -\cos x$$

$$\int dx \sin^2 x = \frac{x}{2} - \frac{1}{4} \sin 2x$$

$$\int dx \sin x \cos x = -\frac{1}{4} \cos 2x$$

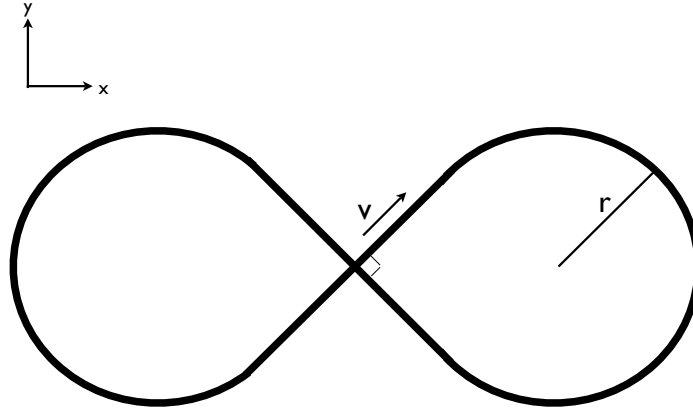
$$\int dx \tan x = -\ln |\cos x|$$

1. Short answer questions (20 points)

These questions can all be answered with at most a few lines of text or a very short calculation.

- (a) (1 point) Define *proper acceleration*.
- (b) (3 points) State Newton's three laws, in their correct special relativistic formulation.
- (c) (3 points) Give three examples of four vectors.
- (d) (3 points) How fast can you run? How fast are you allowed to drive on major interstates? When you ask a question during an 8.20 lecture, how fast does your question travel to my ears? Give each of your answers in terms of a fraction of the speed of light.
- (e) (2 points) If c is the characteristic scale of velocity in the universe, why does everything around us on earth (except for light) travel so much slower than c ?
- (f) (1 point) State the postulates on which Einstein based the special theory of relativity.
- (g) (2 points) State the postulate on which Einstein based the general theory of relativity.
- (h) (1 point) Define *four vector*.
- (i) (2 points) State two experimental results that were important in the acceptance of special relativity as a correct theory of Nature.
- (j) (2 points) State two experimental results that were important in the acceptance of general relativity as a correct theory of Nature.

2. Muons in a figure 8 (20 points)



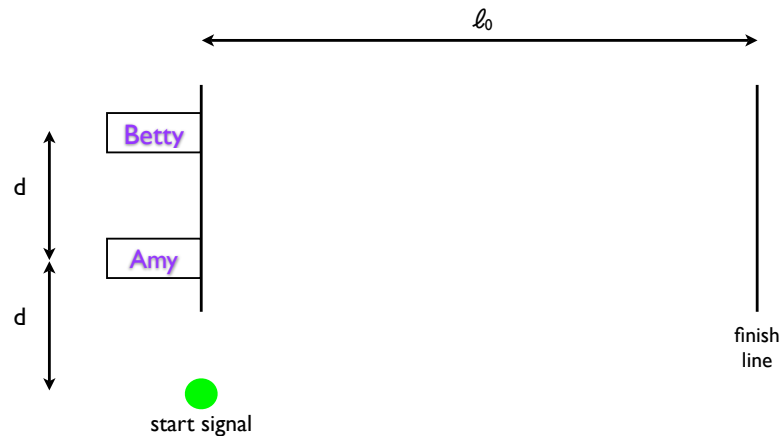
A muon with mass m traverses a path in the form of a figure 8, as shown in the figure above. This figure 8 is formed by two three-quarter circles (with radius r) and two straight sections that cross at a right angle. The muon travels around this path with constant speed $v = \beta c$. The path shown is at rest in the frame of the earth, which we will call Σ . Let the origin of Σ coincide with the center of the figure 8. Let Σ' denote the instantaneous rest frame of the muon. At time $t = 0$ (as measured in Σ), the muon is at the center of the figure 8, traveling with velocity $\vec{v} = v(\frac{1}{\sqrt{2}}\hat{x} + \frac{1}{\sqrt{2}}\hat{y})$.

[Note that the questions for this problem extend to the following page.]

- (2 points) As the muon is traversing the curved sections, what must its acceleration be, as measured in the frame Σ ?
- (2 points) As the muon is traversing the curved sections, what must its acceleration be, as measured in its instantaneous rest frame Σ' ?
- (4 points) Draw the muon's world line on a ct versus x plane (ignore y to make drawing easier), as measured in the frame Σ , for one full lap (from the center of the figure 8 at $t = 0$, around the right loop, then around the left loop, and back to the center of the figure 8). Label relevant values on the ct and x axes. [Hint: Draw this picture big to give yourself enough room.]
- (2 points) Plot the relativistic energy E of the particle (as measured in Σ) versus time (as measured in Σ) for one full lap. Label any important values on the vertical and horizontal axes.
- (2 points) Plot the magnitude of the relativistic momentum $|\vec{p}|$ (as measured in Σ) versus time (as measured in Σ) for one full lap. Label any important values on the vertical and horizontal axes.
- (2 points) What is the muon's energy E' and momentum \vec{p}' in its instantaneous rest frame Σ' ?
- (3 points) How much time (as measured in Σ) does it take the muon to go around one full lap?

- (h) (3 points) How much time (as measured in the sequence of instantaneous rest frames of the muon) does it take the muon to go around one full lap?
- (i) (2 points) Let $v = (1 - 10^{-6})c$ and $r = 100$ m. The mass of the muon is $m = 106 \text{ MeV} = 106 \times 10^6 \text{ eV}$, and the lifetime of the muon is $\tau_\mu = 2.2 \mu\text{s}$. On average (and to an accuracy of say 10%, to encourage estimation), how many times will the muon be able to go around before it decays?

3. Space race (20 points)



Two space car drivers (Amy and Betty) have agreed to a race. They line up on the starting line, as shown in the figure above (drawn in an inertial frame Σ). To prevent accidents, the racing lanes are $d = 1$ light-second apart (as measured in Σ). The starting signal is a light signal sent from the position shown in the figure, a distance $d = 1$ light-second away from Amy, at time $t = 0$ (as measured in Σ). Amy takes off as soon as she receives the starting signal, and Betty takes off as soon as she receives the starting signal. [Note that Amy and Betty do not start at the same time, as measured in Σ .] Amy has an excellent space car, able to accelerate at $a'_A = 1.5 \times 10^6$ m/s² (as measured in the instantaneous rest frame of her car). Betty has an even better space car, able to accelerate at $a'_B > a'_A$ (as measured in the instantaneous rest frame of her car). For both Amy and Betty, the total mass of car plus driver is m ; assume m stays constant during the race. Let the starting line correspond to $x = 0$ in the frame Σ . The finish line is exactly $\ell = 1$ light-minute away from the starting line (as measured in Σ). The finish line is thus at $x = \ell$. Leave your answers in terms of symbols (rather than numbers).

[Note that the questions for this problem extend to the following page.]

- (1 point) Write an analytic expression for $x_A(t)$, the position of Amy's car (as measured in Σ) as a function of time t (as measured in Σ).
- (5 points) Draw the worldlines for both Amy and Betty on a single ct versus x plane, as measured in Σ . Label relevant points on both the vertical and horizontal axes.
- (2 points) At what time (as measured in Σ) does Amy cross the finish line?
- (2 points) What is Amy's speed (as measured in Σ) when she crosses the finish line?
- (2 points) Suppose the parameters of this problem (d , ℓ , m , a'_A , and a'_B) had different values. How could you quickly determine whether a non-relativistic treatment of this problem would give an answer accurate to better than a few percent? Write your answer as an inequality involving these parameters.

- (f) (2 points) Just as she crosses the finish line, what is Amy's energy and momentum (as measured in Σ)?
- (g) (2 points) From the information provided, can you tell who wins? If so, who? If not, what more would you need to know?
- (h) (2 points) If the distance to the finish line were increased, would this favor Amy or Betty?
- (i) (2 points) Is it possible in principle for Betty's car to accelerate faster than $3 \times 10^8 \text{ m/s}^2$? Justify your answer.

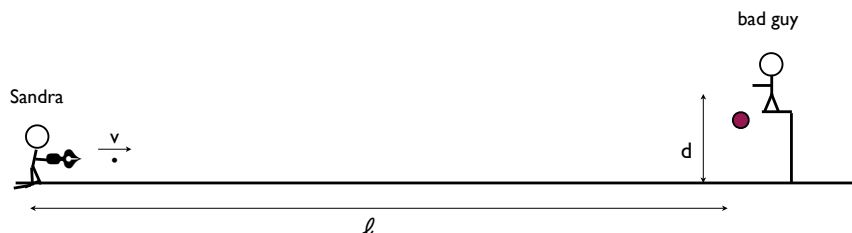
4. Cosmic rays (20 points)

Consider the process of a very high-energy proton (p) absorbing a photon (γ), and producing a new particle Δ^+ . In symbols, the process is thus $p + \gamma \rightarrow \Delta^+$. The mass of the Δ^+ is $m_{\Delta}c^2 = 1.2 \text{ GeV} = 1.2 \times 10^9 \text{ eV}$, and the mass of the proton is $m_p c^2 = 0.9 \text{ GeV} = 0.9 \times 10^9 \text{ eV}$. In the rest frame of the earth (which we will call Σ), the photon of interest to us has energy $E_{\gamma} \approx 6 \times 10^{-4} \text{ eV}$. This energy E_{γ} is typical of the photons left from the Big Bang, permeating the universe in the form of the Cosmic Microwave Background Radiation. Consider this to be a one-dimensional collision problem, with the proton and photon initially racing towards each other. [Hint: In this problem recall from quantum mechanics that $E_{\gamma} = h\nu_{\gamma}$, where h is Planck's constant, and the frequency ν_{γ} gets Doppler shifted in a moving frame.] Leave all your answers in terms of symbols (rather than numbers), except for those questions that explicitly ask for numbers.

- (a) (5 points) Go into the rest frame of the produced Δ^+ , which we will call Σ' . What does the energy E'_p of the proton need to be (as measured in Σ') for the process $p + \gamma \rightarrow \Delta^+$ to occur?
- (b) (3 points) For this proton energy E'_p , what is the corresponding energy E'_{γ} of the photon (as measured in Σ')?
- (c) (3 points) How fast is Σ' moving with respect to Σ ? (In this problem the frame Σ is for all practical purposes defined as the frame in which $E_{\gamma} \approx 6 \times 10^{-4} \text{ eV}$.) [Hints: Recall that Einstein's Nobel Prize in 1921 was actually given for $E_{\gamma} = h\nu_{\gamma}$. You may wish to write $\beta = 1 - \epsilon$, where ϵ is sufficiently small that terms of order ϵ^2 can be neglected.]
- (d) (2 points) What is the Lorentz factor γ corresponding to the velocity you determined in part (c)? What is the numerical value?
- (e) (4 points) Now transform back to the frame Σ . What does the energy E_p of the proton need to be (as measured in Σ) for the process $p + \gamma \rightarrow \Delta^+$ to occur?
- (f) (1 point) What is the numerical value of the energy E_p ? [Hint: You should get a very big number, when expressed in units of electron volts.]
- (g) (2 points) How fast will the resulting Δ^+ particle be traveling (as measured in Σ)?

Your calculation above suggests there should be a limit to the energy of cosmic ray protons that reach us from extragalactic sources. Investigating this limit experimentally is currently one of the hottest topics in particle astrophysics.

5. Saving New York (20 points)



A bad guy has a nuclear bomb, rigged up to detonate on any sudden impact. He is standing on a platform in Times Square. At time $t = 0$ (as measured in the rest frame of Times Square, which we will call Σ), he drops the bomb from a distance $d = 10$ meters above the sidewalk. The bomb falls due to gravity.

Superheroine Sandra is in Albuquerque, $\ell = 3000$ km away. (Pretend the earth is flat in this problem, and assume there is nothing of interest between the East Coast and Albuquerque.) As soon as the bad guy drops the bomb, a signal is sent at the speed of light from a camera in Times Square noting the drop in a straight line to Superheroine Sandra. As soon as Sandra receives the signal, she quickly (in exactly $t_{\text{crouch}} = 1$ second) crouches to the appropriate height and fires a special gun with a superbullet that travels horizontally at a very fast speed $v = \beta c = 0.99c$ toward the falling bomb. The superbullet is specially designed to cause premature detonation of the bomb (before critical density is achieved), resulting in only a small explosion rather than one that would destroy much of New York City.

[Note that the questions for this problem extend to the following page.]

- (1 point) At what time t_1 (as measured in Σ) does Sandra receive the signal that the bomb has been dropped?
- (1 point) As soon as Sandra receives the signal, she crouches to the appropriate height (which takes her $t_{\text{crouch}} = 1$ second, as measured in Σ) and fires her gun. At what time t_2 (as measured in Σ) does she fire her gun?
- (1 point) At what time t_3 (as measured in Σ) does her bullet hit the bomb?
- (1 point) From what height y (as measured in Σ) should Sandra fire her gun so as to hit the bomb? (Assume the mass of the bullet is negligibly small, and that the Earth is flat, so that the bullet reaches the bomb at the same height y above the ground as the height y it was fired from.) [Hint: To simplify the numbers, feel free to round. We are looking for answers accurate to $\approx 10\%$.]
- (4 points) Draw the worldlines of the bullet and the bomb in a ct versus x axis in the rest frame Σ . Label Times Square and Albuquerque on the x axis, and times t_1 , t_2 , and t_3 on the vertical axis.
- (2 point) Let Σ' be the inertial frame in which the bullet is at rest during its flight. Let the spatial origins of Σ and Σ' coincide at the spacetime point at which

Sandra fires the gun. Draw the Σ' axes ct' and x' as seen in the Σ coordinate system. [Hint: Your resulting figure should have four lines, corresponding to the axes ct' , x' , ct , and x .]

- (g) (4 points) Draw the worldlines of the bullet and the bomb in a ct' versus x' plane in the rest frame Σ' .
- (h) (2 points) At what time t'_3 (as measured in Σ') does the bullet hit the bomb?
- (i) (2 points) Write an expression for the vertical distance (as measured in Σ') the bomb falls during this time t' , and from this determine an expression for the height y' above the ground (as measured in Σ') at which the bullet hit the bomb.
- (j) (2 points) Does $y = y'$? Justify your result in words.

**Have a great spring semester!
See you in the infinite corridor.**