

Massachusetts Institute of Technology
Physics Department

Physics 8.20
Special Relativity
Room 4-370

IAP 2008
February 1, 2008
2:30–5:00 pm

Final Solutions

Problem 1 [20 points] Short Answer Questions

- (a) (1 point) Proper acceleration is the acceleration as measured in the object's instantaneous reference frame.
- (b) (3 points)
- In an inertial frame, objects in motion with velocity \vec{v} continue in motion with velocity \vec{v} unless acted upon by some external force.
 - $\vec{F} = \frac{d\vec{p}}{dt}$
 - For every action there is an equal and opposite reaction.
- (c) (3 points) A spacetime point (ct, \vec{x}) ; the 4-velocity $U = \gamma(u)(c, \vec{u})$, and the 4-momentum $p = (E/c, \vec{p})$.
- (d) (3 points)
- $10 \text{ m/s} \approx 3 \times 10^{-7}c$
 - $60 \text{ mph} \approx 30 \text{ m/s} \approx 10^{-7}c$
 - $300 \text{ m/s} \approx 10^{-6}c$
- (e) (2 points) Getting an object with finite mass m to move with velocity \vec{v} with respect to the earth requires giving it a kinetic energy of $KE = (\gamma - 1)mc^2$, assuming the object starts at rest with respect to the earth. Getting a small object with $m = 1$ gram (say) going at $\beta = v/c = 0.5$ thus requires an energy of $2/\sqrt{3}mc^2 \approx 10^{14}$ Joules. This is an extremely large amount of energy, given that most of the energy available for such a purpose is given by chemical (electronic) bonds.
- (f) (1 point)
- The laws of physics take the same form in all inertial frames.
 - The speed of light is the same in all inertial frames.
- (g) (2 points) Uniform acceleration is equivalent to a uniform gravitational field.

- (h) (1 point) A four vector is a set of four numbers $w = (w_0, w_1, w_2, w_3) = (w_0, \vec{w})$ which transform as follows:
- Under rotations, w_0 does not change, and \vec{w} changes like a spatial 3-vector.
 - Under Lorentz transformations (boosts), $w = (w_0, w_1, w_2, w_3)$ transforms like (ct, x, y, z) .
- (i) (2 points) The Michelson-Morley experiment and stellar aberration (for example)
- (j) (2 points) Pound-Rebka experiment (gravitational redshift), exact equivalent of gravitational and inertial masses (Galileo, Eötvös, Dicke), bending of light by the sun, clocks run slow in a gravitational field (Global Positioning System), or others.

Problem 2 [20 points] Muons in a Figure 8

- (a) The muon's acceleration in the curved sections is

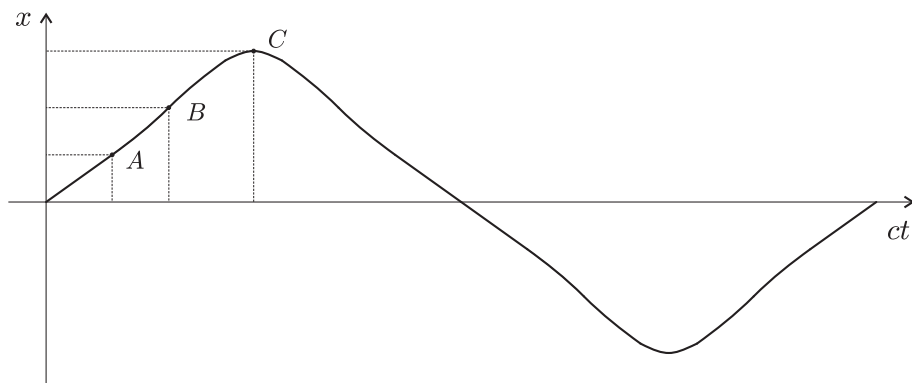
$$\underline{a_{\perp}} = \frac{v^2}{r}, \quad (1)$$

perpendicular to the direction of motion.

- (b) Transforming into the instantaneous rest frame Σ' of the muon, it has nonzero acceleration only in the direction perpendicular to the x' axis (the direction of motion of the Σ' frame), so we obtain it by transforming from the frame Σ as

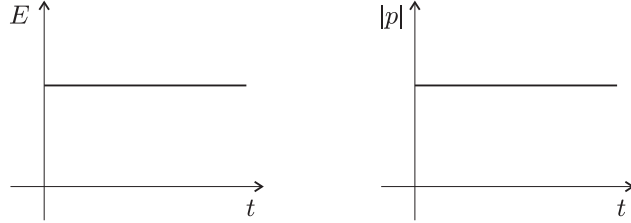
$$a'_{\perp} = \underline{\gamma^2 a_{\perp}}. \quad (2)$$

- (c) We plot the muon's worldline in the ct, x plane to fit on the page (-:



Starting at the center of the loop, the muon traverses x -distance $x_A = \frac{r}{\sqrt{2}}$ with x -velocity $\frac{v}{\sqrt{2}}$, before it hits the curve at point A at time $t_A = \frac{r}{v}$. Its x -component of the velocity then increases, until it becomes v at point B (top of the loop), at position $x_B = 2x_A$, at time $t_B = \frac{r(1+\pi/4)}{v}$. The x -velocity then decreases, until at point C (the rightmost point of the loop) it becomes zero, as the muon is traveling down in the y -direction. This happens at $x_C = r(1 + 2\sqrt{2})$ and at $t_C = T/4$. The rest of the picture follows from symmetry.

- (d) As the magnitude of the muon's velocity does not change, the muon's energy does not change with time, $E = \gamma m_\mu c^2$.



- (e) The magnitude of the muon's momentum is also constant, $|\vec{p}| = \gamma m_\mu v$.
(f) Let us transform E and \vec{p} to the frame Σ' . In the rest frame of the particle, the energy is simply the rest energy $E' = m_\mu c^2$ and the momentum \vec{p} is zero.
(g) The length of the lap is

$$L = 2 \times \frac{3}{4} \times 2\pi r + 2 \times 2r = (3\pi + 4)r. \quad (3)$$

The time to go around one full lap is thus

$$T = \frac{L}{v} = \frac{(3\pi + 4)r}{v}. \quad (4)$$

- (h) In the rest frame of the muon, as its speed doesn't change, the time flows according to time-dilation,

$$T' = \frac{T}{\gamma} = \frac{(3\pi + 4)r}{\gamma v}. \quad (5)$$

- (i) Given $v = (1 - x)c$ with $x = 10^{-6}$, we can approximate

$$\frac{v^2}{c^2} = 1 - 2x + x^2 \approx 1 - 2x, \quad (6)$$

$$\frac{1}{\gamma} = \sqrt{1 - \frac{v^2}{c^2}} \approx \sqrt{2x} \approx 1.4 \times 10^{-3}. \quad (7)$$

An estimate for T gives us

$$T \approx \frac{(3 \times 3.14 + 4) \times 100 \text{ m}}{3 \times 10^8 \text{ ms}^{-1}} \approx 4.5 \times 10^{-6} \text{ s}. \quad (8)$$

Therefore, in the muon's frame,

$$T' = \frac{T}{\gamma} \approx 4.5 \times 10^{-6} \times 1.4 \times 10^{-3} \text{ s} \approx 6.3 \times 10^{-9} \text{ s}. \quad (9)$$

The number of times the muon will traverse the loop is thus roughly

$$n = \frac{\tau_\mu}{T'} \approx \frac{2.2 \times 10^{-6}}{6.3 \times 10^{-9}} \approx \underline{350}. \quad (10)$$

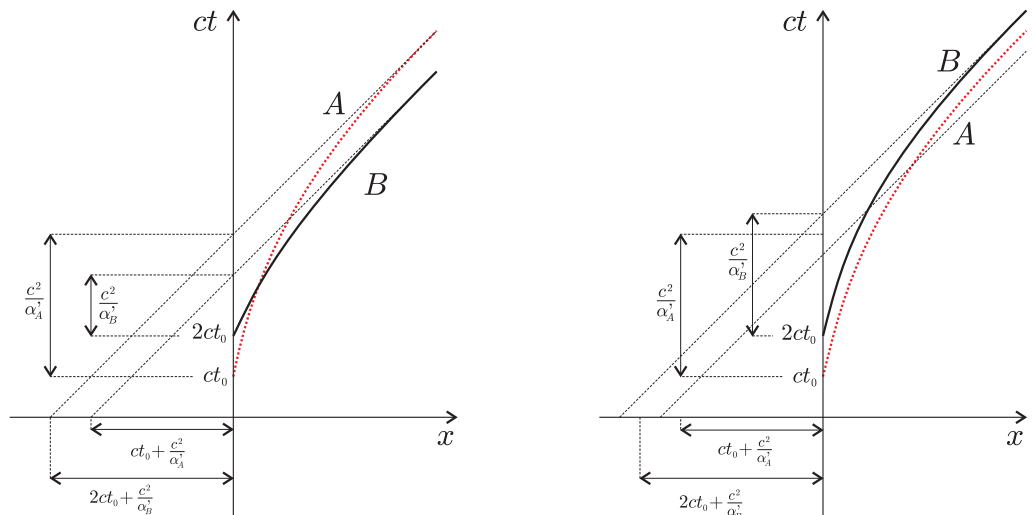
Problem 3 [20 points] Space Race

- (a) The position of Amy's car as measured in Σ is $x_A(t) = 0$ for time $t < t_0 = \frac{d}{c}$ (before she receives the start signal) and then

$$x_A(t) = \frac{c^2}{\alpha'_A} \left(\sqrt{1 + \left(\frac{\alpha'_A (t - t_0)}{c} \right)^2} - 1 \right), \quad (11)$$

for time $t > t_0 = \frac{d}{c}$.

- (b) Let us draw the worldlines for Amy and Betty. As we don't know the exact placing of the asymptotes, we can get two different pictures. Note that in the right one, Betty can't possibly ever catch up! See the answer to h) below for an analysis of these two possibilities.



(c) We obtain this from setting $x_A(t_A) = l_0$, giving us

$$t_A = t_0 + \frac{c}{\alpha'_A} \sqrt{\left(\frac{l_0 \alpha'_A}{c^2} + 1\right)^2 - 1}. \quad (12)$$

(d) Her speed as she crosses the finish line is

$$v_A(t_A) = \frac{\alpha'_A(t_A - t_0)}{\sqrt{1 + \left(\frac{\alpha'_A(t_A - t_0)}{c}\right)^2}} = \frac{\left(\frac{\alpha'_A(t_A - t_0)}{\frac{l_0 \alpha'_A}{c} + c}\right)}{c}. \quad (13)$$

(e) Let us try to treat the problem classically. Using $x = \frac{1}{2}\alpha'_A t^2$, we find that Amy reaches the finish line at time

$$t_{cl} = \sqrt{\frac{2l_0}{\alpha'_A}}, \quad (14)$$

and her final velocity is then

$$v_{cl} = \alpha'_A t_{cl} = \sqrt{2l_0 \alpha'_A}. \quad (15)$$

If this final velocity is much less than c , the classical approximation would hold. As an inequality, we would write

$$\sqrt{2l_0 \alpha'_A} \ll c. \quad (16)$$

(f) As Amy crosses the finish line, her energy is

$$E_A(t_A) = \gamma m c^2, \quad (17)$$

and her final momentum is

$$p_A(t_A) = \gamma m v_A, \quad (18)$$

with γ corresponding to the velocity v_A .

(g) We can't tell who will win, as we don't know the value of α'_B . See the pictures in b) showing both possibilities. The value of α'_A and t_0 are such that they allow for the result to go either way, depending on the value of α'_B , as we'll see below.

(h) With an increased distance to the finish line, the person with the larger acceleration (Betty) would seem to be favored, as she would have more time to catch up and beat Amy, who has a small head start on her.

However, let us look at the problem more closely. Recall the graphs we plotted in b). The distance of the intercept of the asymptote of $x(t)$ on the ct axis from the point where the motion started is $\frac{c^2}{\alpha}$. At a large time T (or equivalently, far from the start), Amy, who starts at time $t_0 = \frac{d}{c}$, will be at

$$x_A \approx \frac{c^2}{\alpha'_A} \left(\frac{\alpha'_A(T - t_0)}{c} - 1 \right) \approx c(T - t_0) - \frac{c^2}{\alpha'_A}. \quad (19)$$

Betty, starting at time $2t_0$ will be at

$$x_B \approx \frac{c^2}{\alpha'_B} \left(\frac{\alpha'_B(T - 2t_0)}{c} - 1 \right) \approx c(T - t_0) - ct_0 - \frac{c^2}{\alpha'_B}. \quad (20)$$

Who will be ahead then? Let us look at the sign of

$$\Delta = x_A - x_B \approx c \left(t_0 - \frac{c}{\alpha'_A} + \frac{c}{\alpha'_B} \right). \quad (21)$$

If the acceleration α'_A is large (or t_0 is large) such that

$$t_0 > \frac{c}{\alpha'_A}, \quad (22)$$

the value of Δ is necessarily positive and Amy wins independently of what car Betty drives. Pictorially, this corresponds to the asymptote of Amy's motion intersecting the t axis below the point where Betty starts. We are given the values for t_0 and α'_A , and plugging them in reveals that $t_0 < \frac{c}{\alpha'_A}$. The race could thus go both ways, depending on how big Betty's acceleration α'_B is.

- (i) Why not? Imagine a huge instantaneous acceleration. The object would accelerate to nearly the speed of light really fast, but its acceleration (in the lab frame) would drop like α/γ^3 in the process. The velocity of the object again could not get larger than c .

Problem 4 [20 points] Cosmic Rays

- (a) Let us write the reaction in terms of the four momenta in the CM frame (the frame of the produced Δ^+).

$$p_p + p_\gamma = p_\Delta. \quad (23)$$

It would prove useful to rearrange it so that we have the photon and the proton on different sides of the equation. Let us then write

$$p_\gamma = p_\Delta - p_p, \quad (24)$$

$$(p_\gamma)^2 = (p_\Delta - p_p)^2, \quad (25)$$

$$p_\gamma^2 = p_\Delta^2 + p_p^2 - 2(p_\Delta \cdot p_p) \quad (26)$$

$$0 = m_\Delta^2 c^2 + m_p^2 c^2 - 2 \left(\frac{m_\Delta c^2}{c} \frac{E'_p}{c} - 0 \right), \quad (27)$$

$$2m_\Delta E'_p = (m_\Delta^2 + m_p^2) c^2 \quad (28)$$

$$E'_p = \frac{1}{2} \left(m_\Delta + \frac{m_p^2}{m_\Delta} \right) c^2. \quad (29)$$

The numerical value of E'_p is

$$E'_p = \frac{1}{2} \left(1.2 + \frac{0.9^2}{1.2} \right) \text{ GeV} \approx 0.94 \text{ GeV}. \quad (30)$$

(b) The corresponding energy of the photon is

$$E'_\gamma = m_\Delta c^2 - E'_p = \frac{1}{2} \left(m_\Delta - \frac{m_p^2}{m_\Delta} \right) c^2. \quad (31)$$

Its numerical value of E'_p is

$$E'_\gamma = \frac{1}{2} \left(1.2 - \frac{0.9^2}{1.2} \right) \text{ GeV} \approx 0.26 \text{ GeV}. \quad (32)$$

(c) We can determine the speed of the frame Σ' from the energies of the photon in the lab frame and in Σ' . The Doppler effect gives us

$$E'_\gamma = \sqrt{\frac{1+\beta}{1-\beta}} E_\gamma, \quad (33)$$

$$\left(\frac{E'_\gamma}{E_\gamma} \right)^2 = \frac{1+\beta}{1-\beta}, \quad (34)$$

$$\left(\frac{E'_\gamma}{E_\gamma} \right)^2 - \beta \left(\frac{E'_\gamma}{E_\gamma} \right)^2 = 1 + \beta, \quad (35)$$

$$\beta = \frac{\left(\frac{E'_\gamma}{E_\gamma} \right)^2 - 1}{\left(\frac{E'_\gamma}{E_\gamma} \right)^2 + 1}. \quad (36)$$

As the energy E_γ of the CMBR is really small, we expect the ratio $\frac{E'_\gamma}{E_\gamma}$ to be really big, and so to obtain $\beta = 1 - \epsilon$. Let us then rewrite β in the form

$$\beta = 1 - \frac{2}{\left(\frac{E'_\gamma}{E_\gamma} \right)^2 + 1} \approx 1 - 2 \frac{\left(\frac{E_\gamma}{E'_\gamma} \right)^2}{\left(\frac{E'_\gamma}{E_\gamma} \right)^2 + 1}. \quad (37)$$

In fact, the numerical value of the ratio $\frac{E_\gamma}{E'_\gamma}$ is only

$$\frac{E_\gamma}{E'_\gamma} \approx \frac{6 \times 10^{-4} \text{ GeV}}{0.26 \text{ GeV}} \approx 2.3 \times 10^{-12}, \quad (38)$$

giving us

$$\beta \approx 1 - 1.1 \times 10^{-23}. \quad (39)$$

(d) The Lorentz factor γ corresponding to the above $\beta = 1 - \epsilon$ is

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} = \frac{1}{\sqrt{1 - (1 - \epsilon)^2}} = \frac{1}{\sqrt{1 - 1 + 2\epsilon - 2\epsilon^2}} \quad (40)$$

$$\approx \frac{1}{\sqrt{2\epsilon}} = \frac{1}{\sqrt{2}} \left(\frac{E'_\gamma}{E_\gamma} \right), \quad (41)$$

and its numerical value is

$$\gamma \approx 0.71 \times 0.43 \times 10^{12} \approx \underline{3.1 \times 10^{11}}. \quad (42)$$

(e) The energy of the proton transformed back to the Σ frame is

$$E_p = \gamma (E'_p + \beta p'_p c). \quad (43)$$

Let us plug in for the proton's momentum from $p'_p c = \sqrt{E'^2_p - m_p^2 c^4}$, which gives us

$$E_p = \gamma \left(E'_p + \beta \sqrt{E'^2_p - m_p^2 c^4} \right). \quad (44)$$

(f) The numerical value of E_p is

$$E_p \approx 3.1 \times 10^{11} \times \left(0.94 + 1 \times \sqrt{0.94^2 - 0.9^2} \right) \text{ GeV} \approx \underline{3.8 \times 10^{20} \text{ eV}}. \quad (45)$$

(g) As the Δ^+ is stationary in Σ' , its velocity in the lab frame Σ is

$$\underline{v_\Delta = \beta c}, \quad (46)$$

with β from (37).

Problem 5 [20 points] Saving New York

(a) The time for the signal to reach Sandra is

$$t_1 = \frac{\sqrt{d^2 + l^2}}{c} = \frac{\sqrt{100 + 9 \times 10^{12}}}{3 \times 10^8} \text{ s} \approx \frac{3 \times 10^6}{3 \times 10^8} \text{ s} \approx 0.01 \text{ s}. \quad (47)$$

(b) Sandra fires her gun at

$$t_2 = \underline{t_1 + t_{\text{crouch}}} \approx 1.01 \text{ s}. \quad (48)$$

(c) The bullet hits the bomb at time

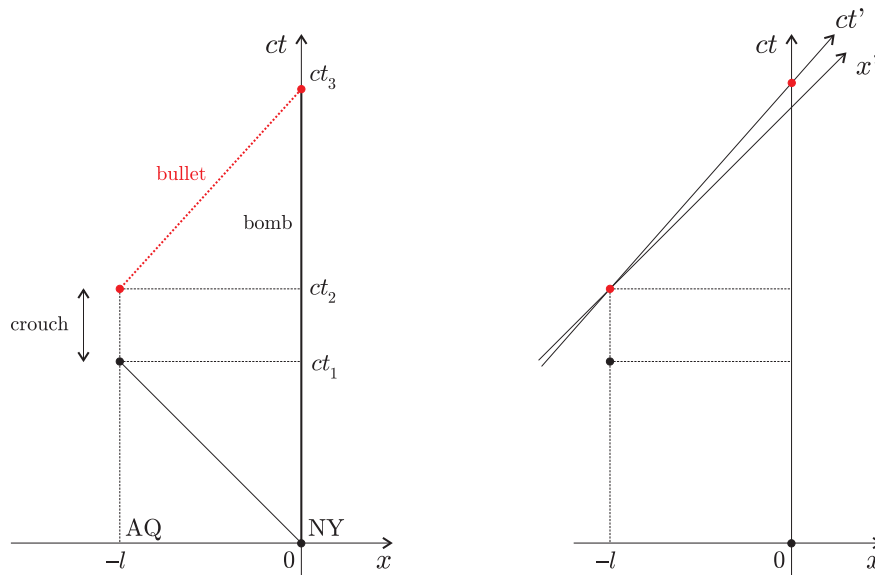
$$t_3 = \frac{t_2 + \frac{l}{v}}{\gamma} \approx \left(1.01 + \frac{3 \times 10^6}{0.99 \times 3 \times 10^8} \right) \text{ s.} \approx 1.02 \text{ s.} \quad (49)$$

(d) At time t_3 , the y-coordinate of the bomb is

$$y = d - \frac{gt_3^2}{2} \approx \left(10 - \frac{10 \times 1^2}{2} \right) \text{ m} \approx 5 \text{ m.} \quad (50)$$

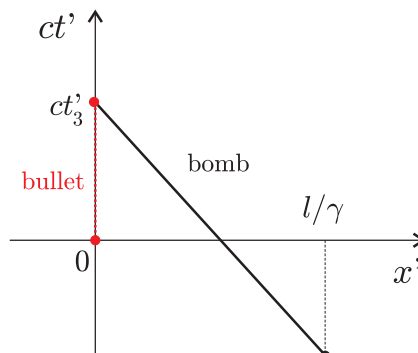
This is the height Sandra should shoot from.

(e) These are the worldlines of the bullet and the bomb in a ct vs. x plot:



(f) The origin $[0, 0]$ of the frame Σ' flies by Sandra at the moment she fires her gun. This event has coordinates $[-l, t_2]$ in the frame Σ . The above figure has the axes for both coordinate systems.

(g) In the frame Σ' , the worldline of the bullet in the ct' vs. x' plot is a vertical line. The bullet is coming in with velocity v as in the picture.



- (h) As noted earlier, the origin of the frame Σ' is shifted by $[-l, t_2]$ with respect to the origin of Σ . Therefore, transforming the event $[0, t_3]$ (the bullet hits the bomb) to the frame Σ' gives us

$$t'_3 = \gamma \left((t_3 - t_2) - \frac{v}{c^2} (0 - (-l)) \right) \quad (51)$$

$$= \gamma \left(\frac{l}{v} - \frac{v}{c^2} l \right) = \gamma \left(1 - \frac{v^2}{c^2} \right) \frac{l}{v} = \frac{1}{\gamma} \left(\frac{l}{v} \right) \quad (52)$$

$$= \sqrt{1 - \beta^2} \left(\frac{l}{v} \right), \quad (53)$$

with the numerical value

$$t'_3 \approx \left(\sqrt{1 - 0.99^2} \times \frac{3 \times 10^6}{0.99 \times 3 \times 10^8} \right) \text{ s} \approx 1.4 \times 10^{-3} \text{ s}. \quad (54)$$

- (i) In the frame Σ' , the bullet falls from height y with vertical acceleration $g' = \frac{g}{\gamma^2}$. The bomb is released from height y at time $t_B = 0$ and is hit at time t_3 . Because these events happen at the same x -coordinate in the Σ frame, we know that after the Lorentz transformation into the moving frame Σ' , the time interval between them will be time-dilated as

$$\Delta t' = t'_3 - t'_B = \gamma(t_3 - t_B) = \gamma t_3. \quad (55)$$

Therefore, the distance the bomb has fallen in the frame Σ' is

$$\Delta y' = \frac{g'(\Delta t')^2}{2} = \frac{\frac{g}{\gamma^2} (\gamma t_3)^2}{2} = \frac{g t_3^2}{2}, \quad (56)$$

which is the same distance it has fallen in the frame Σ . Therefore, the height $y' = y$ from (50), as expected.

- (j) The height y' had to be equal to y , as the y coordinate is unchanged under the Lorentz transformation.