Massachusetts Institute of Technology
Physics Department

Physics 8.20 | IAP 2003
Special Relativity | January 21, 2003
Room 2-190 | 7:30–9:30 pm

Midterm

Your name: ______________________________

Instructions

- **You have two hours for this test. Papers will be picked up at 9:30 pm sharp.**
- **The exam is scored on a basis of 100 points.**
- **Please do ALL your work on the pages of the exam. There are extra pages stapled at the end if you run out of space.**
- **Please remember to put your name on each page of the exam.**
- **You may use the single page of notes that you have prepared.**
- **If you have any questions please feel free to ask the proctor(s).**

Information

$c \approx 300\text{km/millisecond}$

Lorentz transformation (along the x-axis) and its inverse

$$\begin{aligned} x' &= \gamma(x - \beta ct) & \qquad x &= \gamma(x' + \beta ct') \\ y' &= y & y &= y' \\ z' &= z & z &= z' \\ ct' &= \gamma(ct - \beta x) & ct &= \gamma(ct' + \beta x') \end{aligned}$$

where $\beta = v/c$, and $\gamma = 1/\sqrt{1-\beta^2}$.

Velocity addition (relative motion along the x-axis):

$$\begin{aligned} u'_x &= \frac{u_x - v}{1 - u_x v/c^2} \\ u'_y &= \frac{u_y}{\gamma(1 - u_x v/c^2)} \\ u'_z &= \frac{u_z}{\gamma(1 - u_x v/c^2)} \end{aligned}$$

Doppler shift

Longitudinal

$$\nu = \sqrt{\frac{1+\beta}{1-\beta}}\ \nu_0$$

Quadratic equation:

$$\begin{aligned} ax^2 + bx + c &= 0 \\ x &= \frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right) \end{aligned}$$

Binomial expansion:

$$(1+a)^b = 1 + ba + \frac{b(b-1)}{2}a^2 + \ldots$$

1. **Problem 1 [27 points] Short Answer Questions**

The questions below should be answered briefly, no more than five lines and no calculations. Please be brief and to-the-point.

(a) [3 points] Define an inertial frame and give an example of a *non*-inertial frame.

An inertial frame is one in which Newton's First Law is observed to hold. Examples of non-inertial frames include: a merry-go-round, the surface of the Earth for motion in the vertical direction, a uniformly accelerating rocket ship, ...

(b) [3 points] Critique the statement: "If two events are simultaneous in one inertial frame then they are simultaneous in all inertial frames."

Not true in the context of special relativity except for events that also occur at the same point in space (in which case they are one event!). Before Einstein it was thought to be self-evident, but the constancy of the speed of light to all observers makes simultaneity relative.

(c) [3 points] Relative to any event, spacetime can be decomposed into *future*, *past*, and *elsewhere*. What, precisely, is the significance of the region called "elsewhere"?

Draw a picture of space time showing future and past light-cones. "Elsewhere" is the region outside the light-cone of the given event. It consists of the region of space time that can neither affect or be affected by the given event.

[Continued on following page]

(d) [3 points] In pre-Einstein physics the concept of an "incompressible fluid" refers to a fluid which cannot be compressed by any force at hand. Is this concept consistent with special relativity? Explain.

No, it is not consistent. It is connected with the failure of rigidity in S.R. If a compressive force is applied to a fluid, the forces required to keep it from compressing can only be transmitted at speeds less than c, so one side of the fluid can begin to move before the other side has time to react.

(e) [3 points] Light is emitted from a star that moves perpendicular to your line of sight at a velocity close to c. Is it redshifted or blueshifted or neither?

It is redshifted. The relativistic Doppler effect redshifts light emitted by a source moving transversely relative to the observer.

(f) [3 points] Define proper time.

Proper time is the time that elapses in the instantaneous rest frame of a process. It can be defined as $\int d\tau = \int \sqrt{dt^2 - |\vec{dx}|^2/c^2}$, *where* τ *is the proper time and* t *and* $\vec{x}$ *refer to the time and space coordinates of the world line of some process according to an inertial observer.*

[Continued on following page]

(g) [3 points] In one sentence summarize the fundamental result of the Michelson-Morley experiment.

The Michelson-Morley experiment was a null experiment. It failed to detect any motion of the Earth relative to the aether over the course of the year.

(h) [3 points] An intense searchlight on the ground sweeps across the sky. A layer of clouds overhead reflects the light from the searchlight beam back to an observer standing next to the search light. Evaluate the following statement: "The spot made by the searchlight as seen by the observer can move faster than the speed of light"

Correct. The moving spot of the searchlight does not correspond to anything moving faster than the speed of light.

2. **Problem 2 [15 points] The Concept of Relativity**

The concept of relativity:

The laws of Mechanics are such that they hold in all inertial frames

was known in mechanics before Einstein, and was based on the "Galilean" transformation,

$$\vec{x}\,' = \vec{x} - \vec{v}t, \qquad t' = t$$

relating the coordinates of inertial observers moving at a relative velocity $\vec{v}$. Which of the following force laws listed below are consistent with this pre-Einstein concept of relativity? Give a *short* explanation.

[Labels 1 and 2 refer to two different bodies. $\vec{x}$, $\vec{v}$, and $\vec{a}$ are the position, velocity, and acceleration, respectively. k, b, and g are positive constants.]

Applying the Galilean transformation we can find out how velocity and acceleration transform from one inertial frame to another:

$$\begin{aligned} \vec{u}\,' &= \vec{u} - \vec{v} \\ \vec{a}\,' &= \vec{a} \end{aligned}$$

(a) $m_1\vec{a}_1 = -k\vec{x}_1$

Non consistent.
If we make a Galilean transformation we get

$$m_1\vec{a}_1\,' = -k(\vec{x}_1\,' + \vec{v}t')$$

which is not of the same form as in the unprimed frame.

(b) $m_1\vec{a}_1 = -b\vec{v}_1$

Not consistent.
If we make a Galilean transformation we get

$$m_1\vec{a}_1\,' = -b(\vec{v}_1\,' + \vec{v})$$

which is not of the same form as in the unprimed frame.

(c) $m_1\vec{a}_1 = -k(\vec{x}_1 - \vec{x}_2)$

Consistent
If we make a Galilean transformation we get

$$m_1\vec{a}_1\,' = -k(\vec{x}_1\,' - \vec{x}_2\,')$$

because the $\vec{v}$ *dependent term cancels out in the difference of* x_1 *and* x_2*. This is just the same form as in the unprimed frame.*

3. **Problem 3 [18 points] Relations between two events**

Two events, A and B, occur on the x-axis and *at the same time* in the frame Σ. In that frame their separation is Δx_0

(a) [4 points] Show that the two events occur at *greater* separation in any other inertial frame, Σ', moving in the $\hat{x}$ direction relative to Σ.

(b) [4 points] Although A and B are simultaneous in Σ, they occur at different times in frames moving with respect to Σ. Show that it is possible to find a frame Σ', (moving along the $\hat{x}$-direction relative to Σ) in which event A occurs *arbitrarily long before* event B.

(c) [10 points] Define the "invariant interval", ΔS^2, and use the Lorentz transformation from Σ to Σ' to prove that it is indeed invariant, ie. prove $\Delta S^2 = \Delta S'^2$. Use the invariant interval to show that in a frame where event A occurs very long before event B, it also occurs very far away from event B.

(a) Make a Lorentz transformation from Σ to any inertial frame Σ' moving along the x direction:

$$\begin{aligned} x'_A &= \gamma\left(x_A - \beta c t_A\right) \\ x'_B &= \gamma\left(x_B - \beta c t_B\right) \end{aligned}$$

But $t_A = t_B$, so $\Delta x' = x'_A - x'_B = \gamma \Delta x_0$. Since $\gamma > 1$ in any frame moving w.r.t. Σ, $\Delta x' > \Delta x_0$.

(b) The time difference between the events in the frame Σ' follows from the Lorentz transformation, $ct' = \gamma(ct - \beta x)$:

$$\Delta t' = t'_A - t'_B = -\gamma\beta(x_A - x_B)/c = -\gamma\beta\Delta x_0/c$$

Since γ ranges from 1 to ∞ as $|\beta|$ ranges from 0 to 1, and since the sign of β can be chosen such that $\Delta t'$ is negative, t'_A can be arranged to occur arbitrarily long before t'_B.

(c) The definition of ΔS^2 is: $\Delta S^2 = c^2 \Delta t^2 - \vec{\Delta} x^2$.
For L.T. along the x axis:

$$\begin{aligned} c^2\Delta t'^2 &= \gamma^2\left(c^2\Delta t^2 + \beta^2\Delta x^2 - 2c\beta\Delta t\Delta x\right) \\ \Delta x'^2 &= \gamma^2\left(\Delta x^2 + \beta^2 c^2\Delta t^2 - 2c\beta\Delta t\Delta x\right) \\ \Delta y' &= \Delta y \\ \Delta z' &= \Delta z \end{aligned}$$

Combining these equations we find

$$c^2\Delta t'^2 - \Delta x'^2 - \Delta y'^2 - \Delta z'^2 = \gamma^2(1-\beta^2)(c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2)$$

since $\gamma^2(1-\beta^2) = 1$, we find that the interval is invariant.
Finally since $\Delta S^2 = -\Delta x_0^2$ in the original frame, Σ, we see that $c^2\Delta t'^2 - \Delta x'^2 = -\Delta x_0^2$ in any other frame. If $\Delta t'^2 \to \infty$, this requires that $\Delta x'^2 \to \infty$ also.

4. **Problem 4 [20 points] Particle decay**

   A ρ-meson decays into two pions. In the rest frame of the ρ the pions are emitted (back-to-back), each with speed 0.7c.

   (a) [6 points] What is the speed of one pion relative to the other?

   (b) [7 points] Suppose the decay is observed in a frame, which we'll call the "lab frame", where the initial ρ has speed $0.9c$ before it decays. In the lab frame, what is the greatest speed that one of the pions can have?

   (c) [7 points] Same conditions as (b). What is the least speed a pion can have?

(a) Use velocity addition:

$$u' = \frac{u+v}{1+uv/c^2}$$

with $u = v = 0.7c$, so $u' = 1.4c/1.49 = 0.94c$.

(b) The pion will have the greatest speed when it is emitted parallel to the direction of motion of the ρ. Then their speeds will add. As observed in the lab:

$$u_{\max} = \frac{0.9c + 0.7c}{1 + (0.9)(0.7)} = 1.6c/1.63 = 0.98c$$

(c) The pion will have the least spped when it is emitted antiparallel to the direction of motion of the ρ. Their speeds will subtract. As observed in the lab:

$$u_{\min} = \frac{0.9c - 0.7c}{1 - (0.9)(0.7)} = 0.2c/.37 = 0.54c$$

5. **Problem 4 [20 points] Trip to a nearby planet**

   Suppose a very interesting planet has been discovered near a star 50 light years from the Earth. In the year 2050 NASA decides to send an expedition to explore this planet. The decision is made that the men and women on the expedition should reach the planet after 20 years of their own body time has elapsed.

   (a) [2 points] What should be the speed at which the expedition's ship travels from Earth to the planet? [You may assume constant speed and ignore initial acceleration and deceleration.]

   (b) [3 points] How long does the expedition appear to take according to the NASA observers on Earth?

   (c) [5 points] The ship carries radio transmitters tuned to the frequency 90 megaHertz (9×10^7 cycles per second) that they use to send messages back to earth. What frequency should the NASA base station tune to in order to receive these signals?

   (d) [10 points] When the ship reaches the half way point, the astronauts get impatient and send out an advanced party. The advanced party leaves the ship with a relative velocity of $0.99c$. [Again ignore acceleration and deceleration.] How many years have passed since the original departure from Earth *according to the astronauts in the advanced party* when they reach the planet?

(a) We need a time dilation factor $\gamma = 5/2$. *Since*

$$\begin{aligned} \gamma^2 &= \frac{1}{1-\beta^2} \\ \beta &= \sqrt{1-\frac{1}{\gamma^2}} \end{aligned}$$

we get $\beta = 0.9165$.

(b) To travel 50 lyr. at $\beta = 0.9165$ *takes 54.55 yrs.*

(c) This is a Doppler shift problem.

$$\nu = \sqrt{\frac{1-\beta}{1+\beta}} = \sqrt{\frac{1-0.9165}{1+0.9165}} \times 9\times10^7 = 18.55 \textit{ megaHertz.}$$

(d) At the time the advanced party leaves $\Delta\tau = 10$ *years of proper time has passed for the astronauts. The advanced party travels with a speed*

$$u = \frac{0.9165c + 0.99c}{1+(0.99)(0.9165)} = 0.999562c$$

which yields a time dilation factor, $\gamma = \frac{1}{\sqrt{1-\beta^2}} = 33.8$. *At a speed so close to the speed of light, the party takes very close to 10 years according to the Earth based observer to reach the planet. This corresponds to* $10/33.8 = 0.296$ *years of time for the astronauts. Thus, they reach the planet after 10.296 years of proper time has passed.*