Massachusetts Institute of Technology
Physics Department

Physics 8.20 | IAP 2008
Special Relativity | January 22, 2008
Room 4-370 | 7:30–9:30 pm

Midterm Solutions

Problem 1 [25 points] Short Answer

(a)
- Physical laws take the same form in all inertial frames.
- Light propagates with speed c in all inertial frames.

(b) $c = 3.00 \times 10^8\,\mathrm{m.s^{-1}}$.

(c) No.

(d) Yes.

(e)
- The Michelson-Morley experiment. Alternative theory: aether. As the M-M gave a 'no change' result when rotating the apparatus, it was in disagreement with the aether theory.
- The aberration of starlight. Alternative theory: aether drag. As we do see the aberration effect, the aether can not be simply dragged along with the Earth.
- Experiments involving fast moving and fast decaying particles (say τ-leptons). Alternative theory: Gallilean time transformation. The particles do travel farther than the classical theory predicts before they decay.

(f) Nothing (or something involving general-relativistic effects, where time runs differently near massive objects).

(g)
- Proper length: A length measured at the same time in a given frame.
- Proper time: A time interval measured at the same position in a given frame.

(h) The allowed values for γ range from 1 to infinity. The allowed values for β range from -1 to 1. Both are dimensionless.

(i) The invariant interval between two spacetime points is given by

$$\Delta s^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2. \tag{1}$$

Problem 2 [16 points] **Conference travel**

(a) In the rest frame of the Earth, the travel to α-Centauri with speed $\beta = 0.8c$ will take

$$t_E = \frac{d_E}{v} = \frac{4}{0.8}\,\text{yrs} = 5\,\text{yrs}. \tag{2}$$

Therefore, you need to leave 5 years before t_0, Earth time.

(b) On the spaceship, the distance to α-Centauri is Lorentz-contracted as

$$d_T = \frac{d_E}{\gamma} = d_E\sqrt{1-\beta^2}. \tag{3}$$

In the traveler's rest frame, α-Centauri is approaching with speed v. Therefore, the time that the traveler will have to prepare is

$$t_T = \frac{d_T}{v} = \frac{4\,\text{lyr} \times \sqrt{1-0.8^2}}{0.8c} = \frac{4 \times 0.6}{0.8}\,\text{yrs} = \underline{3\,\text{yrs}}. \tag{4}$$

Problem 3 [18 points] **Three events**

(a) The interval between 1 and 2 is

$$\Delta s_{12}^2 = c^2(\Delta t_{12})^2 - (\Delta x_{12})^2 - (\Delta y_{12})^2 - (\Delta z_{12})^2 = 1^2 - (-1)^2 - 1^2 - 0^2 = -1. \tag{5}$$

Events 1 and 2 are thus space-like separated. This means that there is a frame in which they occur at the same time.

(b) The interval between 2 and 3 is again space-like:

$$\Delta s_{23}^2 = 1^2 - 0^2 - (-1)^2 - 1^2 = -1. \tag{6}$$

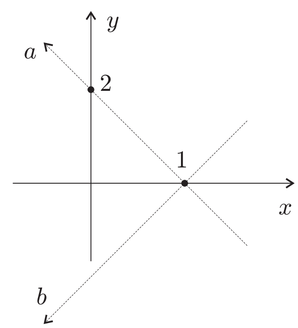
Therefore there is a frame in which they occur at the same time.

(c) The interval between 1 and 3 is time-like,

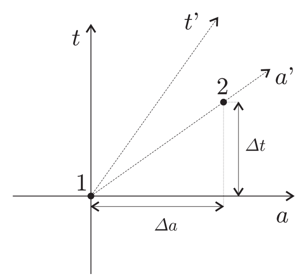
$$\Delta s_{23}^2 = 2^2 - (-1)^2 - 0^2 - 1^2 = 2, \tag{7}$$

i.e. these events are separated by time in any frame. Therefore a frame in which they occur at the same time doesn't exist.

(d) Let us choose events 1 and 2. Let us draw them in the (x, y) plane and define axes (a, b). In this new coordinate system, both events happen on the a axis which brings us back to a familiar 1D problem.



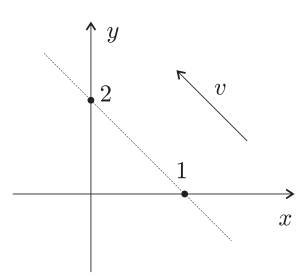
Let us thus redraw the picture in the (a, t) plane.



We want the events to occur at the same time in a moving frame. Its axes must then look like a' and t'. Because the tangent of the angle of the a' axis with the a axis is β, the required velocity of the moving frame in which 1 and 2 occur at the same time is thus

$$\underline{v = c/\sqrt{2}}, \tag{8}$$

in the direction of the axis a, which connects the points where events 1 and 2 occured and points towards 2.



Problem 4 [16 points] Relativistic radar

(a) Using the relativistic Doppler effect formula, the frequency that Y measures is

$$\nu_Y = \sqrt{\frac{c+v}{c-v}}\,\nu_A = \sqrt{\frac{1+\beta}{1-\beta}}\,\nu_A. \tag{9}$$

Here c is the speed of light, v is the velocity with which A approaches Y and ν_A is the frequency of the signal A emits in its own rest frame.

(b) The factor in front of ν_A in the above equation comes from two different reasons. First, we have a factor $\frac{c}{c-v}$ coming from the classical Doppler effect with a moving source (the wavefronts will be closer together as they were emitted at places that closer to each other than if the source was not moving). Because time runs slower for a moving source, the frequency with which it emits the signal will be smaller in the rest frame of the observer. This gives us a second factor $\frac{1}{\gamma} = \frac{1}{c}\sqrt{c^2 - v^2}$. These two multiplied together give the result for the relativistic Doppler effect as it is stated in a).

(c) Let us first expand the relativistic result to first order in $\frac{v}{c}$.

$$\begin{aligned} \frac{\nu_Y^{rel}}{\nu_A} &= \sqrt{\frac{c+v}{c-v}} = \underbrace{\left(1+\frac{v}{c}\right)^{\frac{1}{2}}}_{\approx 1+\frac{1}{2}\frac{v}{c}} \underbrace{\left(1-\frac{v}{c}\right)^{-\frac{1}{2}}}_{\approx 1+\frac{1}{2}\frac{v}{c}} \approx \left(1+\frac{v}{2c}\right)^2 \\ &= 1 + 2\frac{v}{2c} + \frac{v^2}{4c^2} \approx 1 + \frac{v}{c}. \end{aligned} \tag{10}$$

Let us now expand the classical result to first order in $\frac{v}{c}$ and compare it to what we just got:

$$\frac{\nu_Y^{clas}}{\nu_A} = \frac{c}{c-v} = \underbrace{\left(1-\frac{v}{c}\right)^{-1}}_{\approx 1+\frac{v}{c}} \approx 1 + \frac{v}{c}, \tag{11}$$

in agreement with the $\frac{v}{c} \to 0$ limit of the relativistic result.

(d) Let now Y emit a signal with frequency ν_Y. In the rest frame of A, object Y (the source of the signal) is approaching with velocity v. The observer on A will thus receive a signal with frequency

$$\nu_A' = \sqrt{\frac{c+v}{c-v}}\,\nu_Y = \underline{\frac{c+v}{c-v}\,\nu_A}. \tag{12}$$

(e) In the limit $v \to c$, the frequency in d) goes to infinity. Imagine the original source A going not much slower than the signal it emits. Therefore when it 'bounces' off object Y, object A will be coming right behind to receive it all. It will necessarily receive the signal with very high frequency!

Problem 5 [25 points] **Caught speeding**

(a) Using the result of 4d), we have

$$\nu'_A = \frac{1+\beta}{1-\beta}\nu_A, \qquad \rightarrow \qquad \frac{\nu'_A}{\nu_A} = \frac{1+\beta}{1-\beta}. \tag{13}$$

As $\nu'_A/\nu_A = 72/24 = 3$, we obtain

$$3 = \frac{1+\beta}{1-\beta}, \qquad \rightarrow \qquad \beta = 0.5, \tag{14}$$

so you are moving with <u>v=0.5c</u>.

(b) As Beth is traveling towards you, her velocity in your rest frame is

$$v'_B = \frac{v_B + v}{1 + \frac{v_B v}{c^2}} = \frac{\frac{c}{7} + \frac{c}{2}}{1 + \frac{1}{7}\frac{1}{2}} = \frac{2+7}{15}c = \underline{0.6c}. \tag{15}$$

(c) In Beth's rest frame, your velocity is simply $-v'_B$, i.e. <u>0.6c</u> towards her.

(d) The signal that Beth receives comes back to her with frequency

$$\nu'_B = \frac{1+\beta'_B}{1-\beta'_B}\nu_B = \frac{1+0.6}{1-0.6}\nu_B = \frac{1.6}{0.4} \times 24\,\text{GHz} = \underline{96\,\text{GHz}}. \tag{16}$$

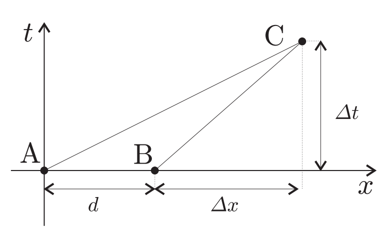
(e) As Deb is coming from behind, in your rest frame her speed is

$$v'_D = \frac{v_D - v}{1 - \frac{v_D v}{c^2}} = \frac{\frac{2}{3}c - \frac{1}{2}c}{1 - \frac{2}{3}\frac{1}{2}} = \frac{\frac{1}{6}}{\frac{2}{3}} = \underline{0.25c}. \tag{17}$$

(f) Deb also sees you approaching with speed $0.25c$, therefore the frequency of the signal she receives is going to be

$$\nu'_D = \frac{1+\beta'_D}{1-\beta'_D}\nu_D = \frac{1+0.25}{1-0.25}\nu_D = \frac{\frac{5}{4}}{\frac{3}{4}} \times 24\,\text{GHz} \approx \underline{40\,\text{GHz}}. \tag{18}$$

(g) At time $t = 0$, Deb is $d = 1\,\text{lmin}$ behind you. Let us draw the situation in the ITA frame.



In the ITA frame, when Deb catches you, her position has to be equal to your position, i.e.

$$v_D \Delta t = d + v\Delta t, \tag{19}$$

which gives us

$$\Delta t = \frac{d}{v_D - v} = \frac{1\,\mathrm{lmin}}{\frac{2}{3}c - \frac{1}{2}c} = \frac{1}{\frac{1}{6}}\,\mathrm{min} = 6\,\mathrm{min}. \tag{20}$$

The distance you traveled in the meantime from B (where you were at $t = 0$) is

$$\Delta x = v\Delta t = 0.5c \times 6\,\mathrm{min} = 3\,\mathrm{lmin}. \tag{21}$$

Using the invariance of the spacetime interval between events B and C (when she catches you), we can determine the time $\Delta t'$ that has passed in your frame between these events. In your frame, the events are separated only by time. Therefore

$$(c\Delta t')^2 = (c\Delta t)^2 - (\Delta x)^2 = (c\Delta t)^2 \left(1 - \beta^2\right), \tag{22}$$

giving us

$$\Delta t' = \Delta t\sqrt{1-\beta^2} = 6\,\mathrm{min} \times \sqrt{1 - \frac{1}{4}} = 3\sqrt{3}\,\mathrm{min} \approx \underline{5.20\,\mathrm{min}}. \tag{23}$$