Physics 8.20 Special Relativity IAP 2008

Problem Set #3 Due Friday January 25

Topics for this period

- Space-time diagrams
- The algebra of Lorentz Transformations
- Paradoxes

Reading

- Resnick, Chs. 1 – 2 and Supplements A and B
- French Chs. 1– 5
- Einstein §1 –17 and Appendix 1
- If you want to read ahead, also read Resnick Ch. 3 and French Ch. 6

Problems

Note: All problems marked **(RH)** *are taken from Resnick and Halliday, Basic Concepts in Relativity.*

1. **Longitudinal and transverse Doppler shifts in terms of wavelengths** (5 points)

   (a) Show that the Doppler shift formulas can be written in the following forms:

$$\begin{aligned}\lambda &= \lambda_0(1-\beta+\frac{1}{2}\beta^2+\ldots) \quad \text{approaching}\\ \lambda &= \lambda_0(1+\beta+\frac{1}{2}\beta^2+\ldots) \quad \text{receeding}\\ \lambda &= \lambda_0(1+\frac{1}{2}\beta^2+\frac{3}{8}\beta^4+\ldots) \quad \text{transverse}\end{aligned}$$

   where λ_0 is the wavelength measured by an observer at rest with respect to the source.

   (b) Derive a similar expression valid through order β^2 when the source makes an angle θ relative to the direction away from the observer.

2. **The Most Distant Galaxy Known:** (3 points)

   One of the most distant galaxies that has been observed is described as "having a redshift of $z = 6.58$". In this problem, we explore what this means. This galaxy was discovered in 2003 by the Subaru Telescope, a facility supported by the National Astronomical Observatory of Japan. You can read about it at the following link: <http://www.naoj.org/Pressrelease/2003/03/>.

Hot hydrogen gas emits light with a particular set of frequencies, referred to as emission lines. The spectrum of emission lines serves as a finger print, allowing an astronomer who observes it to deduce that she is seeing hot hydrogen gas. The "Lyman-α line" is a prominent feature of the hydrogen spectrum which has a wavelength of 121.6 nm, or 1.216×10^{-7} m. This is the wavelength measured in a laboratory experiment, in which both the hot hydrogen and the detection apparatus (the "telescope") are at rest. [Note that this wavelength is deep in the ultraviolet. The human eye is sensitive to light with wavelengths from about 400 nm (violet light) to about 700 nm (red light).]

The Subaru telescope group found a galaxy in which the Lyman-α line has wavelength 922 nm. The Doppler shift is so great that ultraviolet light has become infrared! By convention, the redshift z is defined as $1 + z$ =(observed wavelength/emitted wavelength). In this case, $1 + z = 922/121.6$, yielding $z = 6.58$.

Use the formula for the special relativistic Doppler shift to deduce the relative velocity between the earth and the distant galaxy, assuming that the galaxy is receding directly away from the earth. (This is a good approximation; any transverse velocity would be much smaller than the recession velocity you have calculated.)

Cosmological Aside, beyond the scope of 8.20 (by Krishna Rajagopal) A more complete analysis of the implications of the redshifts of distant galaxies and quasars is beyond the scope of 8.20. For those of you interested, I thought I'd make a few comments.

In the 1930's, Edwin Hubble discovered that the more rapidly a galaxy is receding from us, the farther away from us it is. Furthermore, since light from more distant galaxies takes longer to reach us, we see these distant galaxies as they looked long ago, when the universe was younger than it is today.

You'll just have to take my word for the following implications of a redshift $z = 6.58$, as a derivation requires general relativity. The precise relation between redshift and distance is best stated as follows. Consider two galaxies or quasars — for example, our galaxy and the quasar observed by Subaru — which are separated by a distance $R(t)$. Suppose that $R = R_0$ today. At the time when the light which Fan detected was emitted, R was only $R_0/(1 + z)$. Between that long ago time and today, the distance between any two galaxies in the universe has expanded by a factor $(1 + z)$. Einstein's theory of general relativity (analyzed beyond the level which we will attempt later in 8.20) relates the behavior of $R(t)$ to the density of matter (i.e. galaxies) in the universe. This density is not known well, but is not too far from the "critical density", for which Einstein's equations make the particularly simple prediction that $R(t)$ is proportional to $t^{2/3}$. This means

$$\frac{R(t)}{R_0} = \left(\frac{t}{t_0}\right)^{2/3}$$

where t_0 is the present age of the universe. Putting it all together, when the light from the distant quasar was emitted, the universe was only $t_0/(1 + z)^{3/2}$ years old, or about 1/20 of its present age!

3. Visual appearance of a rod (7 points)

This problem essentially recapitulates the discussion in lecture.

Suppose a rod of rest length ℓ_0 is oriented along the x' axis in the frame S'. The frame S' moves to the right with velocity v along the x-axis when viewed from the frame S. The origins of S and S' coincide at $t = t' = 0$ and their axes are parallel.

An observer sits at the origin of S and watches the rod by looking at the light omitted by it.

(a) What is the *apparent* length of the rod at for times > 0?

(b) How do you reconcile this result with Lorentz Contraction?

(c) Repeat (a) and (b) for velocity $-v$.

4. Lorentz Transformation of Solutions to Maxell's Equations (6 points)

You have previously shown that Maxwell's wave equation

$$\frac{\partial^2}{\partial x^2}E(x,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}E(x,t) = 0 \qquad (1)$$

is invariant under Lorentz transformations and that functions of the form $E(kx - kct)$ satisfy it and correspond to motion to the right with velocity c. Here k is a constant inserted for later convenience. Consider the solution

$$E(x,t) = \sin(kx - kct),$$

which describes a sinusoidal wave with wavelength $\lambda = 2\pi/k$, periond $T = 2\pi/kc$, and frequency $\nu = kc/2\pi$

(a) By direct transformation of $(x - ct)$ to a frame S' moving with speed v relative to S, obtain $E(x', t')$. Show your result for $E(x', t')$ satisfies the wave equation in the S' frame:

$$\frac{\partial^2}{\partial x'^2}E(x',t') - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}E(x',t') = 0$$

(b) The solution in the S' frame, $E(x', t')$, should again be a sinusoidal wave traveling to the right. Determine its wavelength λ', period T', and frequency ν'. Check that $\lambda'\nu'$ yields the correct speed of light. Check that your formulae agree with the Doppler shift.

5. Sequential Lorentz Transformations (6 points) French, problem 5-7, p 160.

6. Lorentz transformations as rotations (5 points) French, problem 3-9, p 87.

7. Space-time diagrams I (10 points) Resnick, Supplementary topic A problem 5, p 200, part a of 12 only, problem 6, p 200, part a of 13 only.

8. Space-time diagrams II (8 points) Resnick, Supplementary topic A problem8, p 200

9. Another version of the famous polevaulter problem (5 points)

The frames Σ' and Σ are related in the standard fashion (as in the previous problem). A (one dimensional) garage is at rest in Σ. Its front end is at $x = 0$; its rear end at $x = L_0$. The garage has doors at both ends. Initially the front door is open and the rear door closed.

A Stacy Dragila (the world's woman's pole vault record holder) runs with velocity v up the x-axis as viewed from Σ. Of course, she is at rest in Σ'. She holds her pole, which is of rest length ℓ_0, parallel to the x-axis. [ℓ_0 is much greater than the rest length of the garage.] The right end of her pole reaches the left end of the garage at $t = t' = 0$. She runs so fast that Lorentz contraction shortens her pole to $\ell = L_0/2$ as viewed in Σ.

In the rest frame of the garage, it is clear that her pole fits into the garage, making the following sequence of events possible:

- **Event A** At the instant that the back end of the pole reaches the front of the garage, the front door of the garage is closed.
- **Event B** At the instant that the front end of the pole reaches the back end of the garage, the back door of the garage is opened.

Note that Event B occurs after Event A in Σ.

(a) What is Dragila's velocity in terms of L_0, ℓ_0 and c?

(b) Find the position and time of Events A and B in Σ.

(c) Now consider the events in Stacy's rest frame. Transform Events A and B to Σ' and show that the back door opens before the front door closes, making it possible for Stacy and the pole to get in and out of the garage without being crushed.

(d) After getting bored opening and closing doors at pre-assigned times during a practice session, the door operators (in Σ) propose another scheme: The front door closer proposes to send a signal to the back door opener as soon as he closes the front door (Event A) telling him to open the back door. Will this signal reach the back door in time to be effective?

10. The airplane falling through the ice (6 points)

In lecture I asserted that a rod (an "airplane") drifting down upon an ice sheet with a hole in it, sees the plane of the ice rotated in its rest frame. The aim of this problem is to establish this effect.

This is a conceptually challenging problem

(a) First, let Σ and Σ' be frames related in the standard fashion (see problem 5) with relative speed v. Suppose a rod at rest in Σ' and of rest length ℓ_0 makes an angle θ' with the x' axis. Find the length of the rod (ℓ) and the angle (θ) that it makes with the x axis as observed in Σ.

(b) Now introduce new (rotated) Cartesian coordinate axes $\tilde{x}'$ and $\tilde{y}'$ in Σ' so that the rod lies along the $\tilde{x}'$ axis. Let the $\tilde{x}$ and $\tilde{y}$ axes be defined to be parallel to the $\tilde{x}'$ and $\tilde{y}'$ axes when the origins of Σ and Σ' coincide. Show that the origin of Σ' has velocity components $(v_{\tilde{x}}, v_{\tilde{y}}) = (v\cos\theta', -v\sin\theta')$ viewed from Σ.

(c) Show that the rod, which lies along the $\tilde{x}'$ coordinate axis in Σ', is observed at an angle to the $\tilde{x}$ coordinate axis in Σ. What is the angle?

11. A Twin Problem (6 points)

French §5, Problem 5-19.

12. Einstein's "clock paradox" (4 points)

[Adapted from R. Resnick *Introduction to Special Relativity*]

Einstein, in his first paper on the special theory of relativity, wrote the following:

> "If one of two synchronous clocks at A is moved in a closed curve with constant velocity until it returns to A, the journey lasting t seconds, then by the clock that has remained at rest the travelled clock on its arrival at A will be $tv^2/2c^2$ seconds slow."

(a) Prove this statement. Note: elsewhere in his paper Einstein stated that this result is an approximation valid for $v \ll c$.

(b) What is the exact statement?

13. Twin Problems (4 points)

[Adapted from R. Resnick *Introduction to Special Relativity*]

This problem refers to Figure B-2 in Resnick. "Bob" is the travelling twin. "Dave" is the stay at home.

(a) In the spacetime diagram of Figure B-2 how far apart are Bob and Dave when Bob turns around?

(b) Suppose that Dave did not know beforehand when Bob was planning to turn around. When (by his own clocks and calendars) would Dave know that Bob had done so?

(c) Suppose that Bob, after noting the passage of three years by his on-board clock, decides not to return to Dave, but simply stops. He compares his on-board clock with one of the local clocks belonging to the synchronized array of stationary clocks fixed in Dave's inertial frame. What will his local clock read? Draw Bob's world line for this new situation on the diagram of Figure B-2.

14. Bob is older than Dave this time! (10 points)

[From R. Resnick & D. Halliday *Basic Concepts in Relativity*]

Bob, once started on his outward journey from Dave, keeps on going at his original uniform speed of $0.8c$. Dave, knowing that Bob was planning to do this, decides, after waiting for three years, to catch up with Bob and to do so in just three additional years.

(a) To what speed must Dave accelerate to do this?

(b) What will be the elapsed time by Bob's clocks when they meet?

(c) How far will they each have travelled when they meet, measured in Dave's original inerital reference frame?

(d) Draw the world lines for Bob and Dave on a spacetime diagram and compare it with Figure B-2. Notice that the present scenario is the mirror image of the one discussed in connection with that figure: there Dave turned out to be four years older than Bob when they reconvened; here Bob will be four years older than Dave.

15. Rapidity (5 points)

The "rapidity" is a useful substitute for speed when a particle is moving very close to the speed of light. The rapidity, η, is defined by

$$\beta = \frac{v}{c} = \tanh\eta.$$

Remember, $\tanh x$ is the "hyperbolic tangent",

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

[Hint: in this problem you will have to know how to find the *inverse hyperbolic tangent*. You can use a calculator, a program like Maple or Mathematica, or you can solve for β in the equation above and express it in terms of logarithms involving η.]

(a) What is the speed of a particle (in units of c) that has $\eta = 1, 2, 3$, and 10?

(b) A proton has rest energy $m_p c^2 = 938$ MeV. What is the total energy of a proton with $\eta = 1, 2, 3,$ 10?

(c) A proton has *kinetic* energy (K.E.$= E - mc^2$) of 1 GeV. What is its rapidity?

16. Rapidities Add (5 points)

All the motion in this problem is collinear — say along the x axis. A particle moves with velocity u' along the x' axis in the frame Σ'. This frame, in turn, moves with velocity v along the x axis in the frame Σ. The particle's velocity observed in Σ is u. This is the standard configuration for velocity addition. In lecture (and in the texts) we derived:

$$u = \frac{u' + v}{1 + u'v/c^2}.$$

Let η, ξ, and ξ' be the rapidities corresponding to v, u, and u', respectively. (So $v = c\tanh\eta$, $u = c\tanh\xi$, and $u' = c\tanh\xi'$).

Show that $\xi = \eta + \xi'$. So in special relativity (parallel) rapidities add.