Physics 8.20 Special Relativity IAP 2008

Problem Set #4 Due Wednesday January 30

Topics for this period

- Acceleration
- Relativistic Dynamics
- Relativistic collisions and decays
- Lorentz transformation of energy and momentum
- Invariants
- Toward general relativity

Reading

- Resnick, Chs. 1 – 3 and Supplements A, B, and C
- French Chs. 1– 7
- Einstein §1 – 32 and Appendix 1

Problems

1. **Proper acceleration** (4 points)

   Note: this problem and the next two ask you to work through the details of results derived in lecture. Let a particle be moving along the x-axis when viewed in the frame Σ. The particle's *proper acceleration*, α, is defined as its acceleration measured in its instantaneous rest frame. Specifically, suppose at time t the particle has velocity v. Then it is instantaneously at rest in the frame Σ' moving with velocity v relative to Σ. Then $\alpha = dv'/dt'$.

   Show that the acceleration observed in Σ is related to α by

   $$\frac{dv}{dt} = \gamma^{-3}(v)\alpha$$

   where $\gamma(v) = 1/\sqrt{1 - v^2/c^2}$ as usual, and v is the instantaneous velocity.

2. **Constant proper acceleration** (5 points)

   Suppose a particle experiences constant proper acceleration, $\alpha = \alpha_0$. Suppose it starts out at rest in Σ at $t = 0$.

(a) Use the result of the previous problem to show that its velocity (as measured in Σ) is given by

$$v(t) = \frac{\alpha_0 t}{\sqrt{1 + (\frac{\alpha_0 t}{c})^2}}$$

(b) Let $\alpha_0 = g = 9.8\text{m/sec}^2$. How long would it take the particle to reach $v = 0.99c$?, $v = 0.999c$? according to an observer in Σ?

(c) How long would it take to reach these speeds according to an observer on the particle?

3. Hyperbolic space travel I (7 points)

(a) Take the result of the previous problem,

$$v(t) = \frac{\alpha_0 t}{\sqrt{1 + (\frac{\alpha_0 t}{c})^2}}$$

and integrate $v(t) = dx/dt$ to obtain an expression for $x(t)$. Take $\alpha = g = 10\text{m/sec}^2$ and show that you reproduce the result obtained in class:

$$X(T) = \sqrt{T^2 + 1} - 1$$

where T is in years and X in light-years.

(b) Likewise confirm the result from lecture that relates the passage of proper time (experienced by the astronauts), τ, and the passage of time in the rest frame from which the rocket originated:

$$T = \sinh \tau$$

— again both times in years.

(c) Now suppose that the space travellers take a more “realistic” journey where they accelerate at g for the first half of the trip and decelerate at g for the second half. Find expressions for

   i. How far can they travel in a time T (measured in the originating frame)?
   ii. How much proper time, τ, passes on a journey that takes T years (as observed in the originating frame).

4. Hyperbolic space travel II (7 points)

Assume that a rocket can produce an acceleration $g_0 = 9.8\text{m/sec}^2$. [As described in lecture, this acceleration is the same as that due to gravity at the surface of the earth. Astronauts will be able to live comfortably in this spaceship, as if in earth’s gravity.) Assume, in addition, that in travelling to any destination the rocket *will accelerate half the way and decelerate during the second half of the journey.*

(a) Calculate the travel time *as measured by the space traveller* to the moon. (Assume the moon is at a distance of 382,000 km.) Compare with the Galilean answer.

(b) Answer the same questions for travel to Neptune, assumed to be at a distance of 4.5×10^9 km.

(c) Answer the same questions for travel to Alpha Centauri, assumed to be at a distance of 4.3 light years away. What is the velocity of the rocket, in the earth's reference frame, at the half way point of the journey?

(d) What is the value of β that would enable a second astronaut, travelling at constant speed, to travel from the earth to Alpha Centauri in the same travel time as that taken by the rocket described above?

5. Hyperbolic space travel III (8 points)

Aliens arrive on Earth and tell us that they have come from very far away. They had been watching light emitted from the primitive Earth and saw that life was evolving. They took a risk and headed off for Earth. Their journey took them a distance of 100,000,000 light-years. After getting our units right, we figured out that only 5 years of proper time elapsed for them during their great journey. Being smart aliens, they travelled hyperbolically — they accelerated at the natural acceleration of gravity on their planet, G, for the first half, and then decelerated for the second half of the journey.

How strong is gravity on the surface of their planet? (Hint: To solve the transcendental equation for G, you could graph both sides as a function of G; or start with a particular value for G and iterate; or use a symbolic manipulation package like Matlab or Mathematica, both available on Athena.)

6. Non-relativistic limit (3 points)

The correct relativistic formula for the energy of a particle of rest mass m and speed v is $E(v) = m\gamma(v)c^2$.

(a) Expand $E(v)$ to order v^4. Identify the first correction to the usual non-relativistic formula for the kinetic energy, $T = \frac{1}{2}mv^2$.

(b) How large is the correction of part a) for the Earth relative to the Sun? In other words, approximately how much heavier is the Earth on account of its motion around the Sun? You will need the Earth's mass and its orbital speed to complete this estimate.

(c) How large a percentage error did you make by using the approximation of part a) instead of the exact result in the case of the Earth's motion around the Sun?

7. Hard work (2 points)

A proton initially moves at $v = 0.99c$ in some reference frame.

(a) How much work must be done on the proton to increase its speed from $0.99c$ to $0.999c$?

(b) How much work must be done to increase its speed from $0.999c$ to $0.9999c$?

[Hint: it is simplest to use energy units throughout this problem. Express the proton's rest mass in terms of its rest energy, $m_pc^2 = 939$ MeV, and use MeV as your units of work.]

8. Solar Power (5 points)

The net result of the fusion reaction that fuels the sun is to turn four protons and two electrons into one helium nucleus,

$$4p^+ + 2e^- \rightarrow \ {}^4He^{++}.$$

Other particles are given off (neutrinos and photons), but you can assume they eventually show up as energy. The masses of the relevant nuclei are as follows:

$m_p = 1.6726 \times 10^{-27}$ kg.
$m_e = 9.1094 \times 10^{-31}$ kg.
$m_{\,^4He} = 6.6419 \times 10^{-27}$ kg.

(a) How much energy is released when a kilogram of protons combines with just enough electrons to fuse completely to form helium?

(b) How many kilograms of methane would you have to burn to produce the same amount of energy? (Go to your favorite source of chemistry information (the Web, perhaps, or the CRC Handbook) and determine the reaction by which methane burns and how much energy is released per gram molecular weight of methane. The usual units are KCal/mole).

9. $\mathbf{E = mc^2}$ (2 points)

Assume the heat capacity of water is equal to 4.2 joules/g-K, and constant as a function of temperature. How much does the mass of a ton of water increase when it is heated from freezing (0°C) to boiling (100°C)? What is the percentage change?

10. Enormous energies (2 points)

Quasars are the nuclei of active galaxies in the early stages of their formation. A typical quasar radiates enery at the rate of 10^{41} watts. At what rate is the mass of this quasar being reduced to supply this energy? Express your answer in solar mass units per year, where one solar mass unit, 1 smu $= 2 \times 10^{30}$ kg, is the mass of our sun.

11. Classical physics and the speed of light (2 points)

(a) How much energy would it take to accelerate an electron to the speed of light according to "classical" (before special relativity) physics?

(b) With this energy what would its actual velocity be?

12. A useful approximation (4 points)

(a) Show that for an extremely relativistic particle, the particle speed u differs from the speed of light c by

$$\Delta u = c - u = \frac{c}{2}\left(\frac{m_0 c^2}{E}\right)^2$$

where m_0 is the rest mass and E is the energy.

(b) Find Δu for electrons produced by

i. MIT's Bates Accelerator Center, where $E = 900$ MeV.

ii. The proposed upgrade at Jeffreson Lab (in Newport News, Virginia), where $E = 12$ GeV.

iii. The Stanford Linear Accelerator Center (in Palo Alto, California), where $E = 50$ Gev.

13. Pressure of light (5 points)

French §6, Problem 6-7, page 201: A photon rocket
French §6, Problem 6-9, page 201: A laser beam

14. Relativistic collisions and decays (17 points)

To the extent possible, these problems should be solved using invariants.

French §6, Problem 6-5, page 200: Three particle decay
French §6, Problem 6-11, page 201: Pion decay
French §6, Problem 6-14, page 202: Impossible processes
French §6, Problem 6-15,part a , page 202: proton collision
French §7, Problem 7-1, page 225: Kaon decay
French §7, Problem 7-2, page 225: Pair production
French §7, Problem 7-4, page 226: Elastic scattering

15. Fixed-target collisions (RH) (5 points)

(a) A proton (with rest mass m) accelerated in a proton synchrotron to a kinetic energy K strikes a second (target) proton at rest in the laboratory. The collision is entirely inelastic in that the rest energy of the two protons, plus all of the kinetic energy consistent with the law of conservation of momentum, is available to generate new particles and to endow them with kinetic energy. Show that the energy available for this purpose is given by

$$\mathcal{E} = 2mc^2\sqrt{1+\frac{K}{2mc^2}}.$$

(b) How much energy is made available when 100-GeV protons are used in this fashion?

(c) What proton energy would be required to make 100 GeV available? This should be compared with the result in the next problem.

16. Center-of-mass collisions (RH) (5 points)

(a) In modern experimental high-energy physics, energetic particles are made to circulate in opposite directions in so-called storage rings and are permitted to collide head-on. In this arrangement, each particle has the same kinetic energy K in the laboratory. The collisions may be viewed as totally inelastic, in that the rest energy of the two colliding protons, plus all the available kinetic energy, can be used to generate new particles and to endow them with kinetic energy. Show that, in contrast to the previous problem, the available energy in this arrangement can be written in the form

$$\mathcal{E} = 2mc^2\left(1+\frac{K}{2mc^2}\right).$$

(b) How much energy is made available when 100-GeV protons are used in this fashion?

(c) What proton energy would be required to make 100 GeV available?

(d) Compare these results with those of the previous problem and comment on the advantages of storage rings.

17. Compton scattering (5 points)

Derive the relationship between scattering angle and wavelength change for Compton scattering.

$$\Delta\lambda = \frac{h}{mc}(1-\cos\theta)$$

Explain what the notation means. What is $\Delta\lambda$? What is $\cos\theta$? [You can derive this using energy and momentum conservation or you can use invariants to make this problem easier.]

18. Production of hyperons at rest (7 points)

The K^- meson and the Λ^0 hyperon are two commonly encountered unstable particles. For example, they are commonly produced in air showers by cosmic rays and several of each have whizzed through you during the time you have been working on this problem set. The reaction

$$K^- + p \to \Lambda^0 + \pi^0$$

can be used to make Λ^0's at rest in the laboratory by scattering K^- mesons off a stationary proton (hydrogen) target.

(a) Find the energy of the incident K^- beam required to just produce Λ^0 hyperons at rest in the lab.

(b) What is the π^0 energy for this "magic" K^- energy?

(c) Check momentum conservation.

(d) Could the process be run the other way? That is, could a π^0 beam (assuming one was available) be used to make a K^+ at rest by the reaction $\pi^0 + p \to \Lambda^0 + K^+$?

The rest energies of all the particles involed are: $m_p c^2 = 939$ MeV, $m_{K^\pm} c^2 = 494$ MeV, $m_{\pi^0} c^2 = 135$ MeV, $m_{\Lambda^0} c^2 = 1116$ MeV.

19. Invariant Product (5 points)

Let

$$A = (a_0, a_x, a_y, a_z) \quad \text{and} \quad B = (b_0, b_x, b_y, b_z)$$

be two four-vectors in frame S and let

$$A' = (a'_0, a'_x, a'_y, a'_z) \quad \text{and} \quad B' = (b'_0, b'_x, b'_y, b'_z)$$

be the corresponding Lorentz transformed four-vectors in frame S'. By writing out the explicit Lorentz transformation for A' and B' in terms of $\sinh(\eta)$ and $\cosh(\eta)$, show that the invariant product

$$A \cdot B = a_0 b_0 - a_x b_x - a_y b_y - a_z b_z$$

is indeed invariant, ie $A \cdot B = A' \cdot B'$.