Physics 8.20 Special Relativity IAP 2008

Problem Set # 2 Solutions

Problems

1. **A Moving Clock** (7 points)

   A clock moves along the x-axis in your reference frame at a speed of $0.80c$ and reads zero as it passes the origin. What time does it read as it passes the 180 meter mark on this axis? How much time has passed in your reference frame?

   In our reference frame, the clock covers a distance $\Delta x = 180m$, in time $\Delta t = \frac{\Delta x}{v} = 0.75\mu s$. The time interval measured in the clock's rest frame is given by the Lorentz transformation:

$$\begin{aligned}\Delta t' &= \gamma(\Delta t - \frac{\beta}{c}\Delta x) \\ &= \gamma \Delta x(\frac{1}{v} - \frac{\beta}{c}) \\ &= \frac{\Delta x}{v\gamma} \\ &= \frac{\Delta t}{\gamma} = .45\mu s,\end{aligned}$$

   which is just the time dialation effect i.e. the clock moves slower in its rest frame.

2. **How fast a ship?** (7 points)

   A spaceship is moving at such a speed in the laboratory frame that its measured length is one third its proper length. How fast is the spaceship moving relative to the laboratory frame?

$$\frac{l_0}{\gamma} = \texttt{measured length} = \frac{1}{3}l_0$$

$$\Rightarrow \gamma = 3$$

$$\Rightarrow \beta = \frac{\sqrt{8}}{3} \approx 0.943$$

$$\Rightarrow v \approx 2.8 \times 10^8 m/s$$

3. String of lights across the desert I (7 points)

A series of lights is arrayed in a straight line across the desert. Neighboring lights are separated by a distance d. They are set up to flash in sequence with an interval τ between neighboring lights (as measured in the rest frame of the lights). An observer, O, travels along the same line at a uniform speed v in the same direction of the wave of flashes.

(a) At what interval do the flashes occur in the rest frame of O?

In the rest frame of the lights, choose a coordinate system with the x-axis along the lights (in the direction of the wave). Then, the space-time separation between two flashes is

$$\Delta t = \tau, \Delta x = d\,.$$

Therefore in O's rest frame,

$$\Delta t' = \gamma\left(\Delta t - \frac{\beta}{c}\Delta x\right) = \gamma\left(\tau - \frac{v}{c^2}d\right)\,.$$

(b) Suppose O travels in the direction opposite the wave. What is the interval in this case?

The interval will now be given by the expression for the interval in part (a) with $v \to -v$. So,

$$\Delta t' = \gamma\left(\tau + \frac{v}{c^2}d\right)\,.$$

(c) For what choices of d, τ, and v do all flashes occur simultaneously in the rest frame of O?

For simultaneous flashes, $\Delta t' = 0$. This could occur when the observer moves in the direction of the wave with

$$\gamma\left(\tau - \frac{v}{c^2}d\right) = 0 \Rightarrow v = \frac{\tau c^2}{d}\,.$$

Of course v cannot exceed c and so the above speed is possible only if the following constraint is satisfied:

$$\frac{d}{\tau} > c\,.$$

A careful reader should pause for a moment and understand the previous equation properly. Suppose $d/\tau < c$, then I can put a sensor on each light so that it receives signal from the previous flash. Now before a flash goes off it would have received a signal from previous flash therefore it would have prior knowledge of the previous flashing. In other words current flash could be a result of previous flash (or at least I can design the experiment that way) that is every flash is causally connected to previous flash. From the previous eqn. we learn that in this case observer O will not be able to see simultaneous flashes for any value of $v < c$ since $d/\tau < c$. If it was possible for observer O to see flashes simultaneously then that will be violation of causality. However if $d/\tau > c$ then I cannot design an experiment in which two flashes are causally connected, hence it is possible to see simultaneous flashes in frame of observer O.

4. String of lights across the desert II (7 points)

In the previous problem you were asked for the times at which the flashed occured in the rest frame of O. Now consider what would be *seen* by the observer O:

Again let the straight line of lights be separated by d and flash in sequence with interval τ. Now compute the interval between the sequential flashes *as seen by* an observer travelling with uniform speed v along the direction of the flashes. [Hint: in this case you must consider not only the Lorentz transform of each flash event, but you must also consider how long it takes a flash to propagate to the observer O.] For what values of d, τ, and v will the flashes *appear to be* simulataneous to O?

Consider the rest frame of the lights. Choose coordinates so that at $t = 0$, the origin is at the light that flashes, the observer is at x_0 (with $x_0 > d$) and the x-axis is along the lights in the direction of the wave. Consider the event of the light from the flash reaching the observer. Its time coordinate, t_1, is determined by

$$ct_1 = x_0 + vt_1 \Rightarrow t_1 = \frac{x_0}{c - v}$$

and its space coordinate is

$$x_1 = x_0 + vt_1 \Rightarrow x_1 = \frac{x_0}{1 - \beta}\,.$$

Then consider the event of the light from the next flash reaching the observer. Its coordinates are determined by

$$c(t_2 - \tau) = x_2 - d\,, x_2 = x_0 + vt_2$$

$$\Rightarrow t_2 = \frac{x_0 + c\tau - d}{c - v}\,, x_2 = \frac{x_0 + v(\tau - d/c)}{1 - \beta}\,.$$

Therefore, the spacetime separation between the two events is

$$\Delta t = \frac{\tau - d/c}{1 - \beta}\,, \Delta x = v\Delta t\,.$$

Transforming this separation into the observer's frame,

$$\Delta t' = \gamma \Delta t\left(1 - \beta^2\right) = \left(\tau - \frac{d}{c}\right)\sqrt{\frac{1 + \beta}{1 - \beta}}\,.$$

So, the flashes will appear to be simultaneous when the observer starts off in front of the wave and the flashes obey the constraint

$$\tau = \frac{d}{c}\,.$$

Note that this is independent of the observer's speed. It is merely the statement that if a light flashes just as the light from the previous flash reaches it, then an observer in front of the wave will see the flashes simultaneously. Note also that if the observer is behind the wave (i.e. $x_0 < 0$) then a similar calculation reveals that the flashes could never be seen simultaneously by the observer.
These results on the simultaneity of the flashes can be understood easily by considering the space-time diagram, but that is left as an exercise.

5. Events in two different frames I (7 points)

Two events, A and B, are observed in two different inertial frames, Σ, and Σ'. Frame Σ' moves along the x axis in Σ. Event A occurs at the spacetime origin in both frames,

$$\begin{aligned} x_A = y_A = z_A = ct_A = 0 \\ x'_A = y'_A = z'_A = ct'_A = 0 \end{aligned}$$

Event B occurs at

$$x_B = 10, \quad y_B = z_B = 0, \quad ct_B = 6$$

(all distances are in meters).

The two events occur *simultaneously* in frame Σ'.

(a) Find the velocity of Σ' with respect to Σ.

$$\begin{aligned} 0 &= \Delta t' \\ \Rightarrow 0 &= \gamma\left(\Delta t - \frac{\beta}{c}\Delta x\right) \\ \Rightarrow \beta &= \frac{c\Delta t}{\Delta x} \\ \Rightarrow v &= 0.6c\,. \end{aligned}$$

Note that $|c\Delta t|$ has to be smaller than $|\Delta x|$ so that $|\beta| < 1$. Such events are called space-like separated and there always exists a frame (Σ' in this case) in which two space-like separated events occur simultaneously.

(b) What is the space separation of the two events in the frame Σ'?

$$\begin{aligned} \Delta x' &= \gamma\,(\Delta x - \beta c\Delta t) \\ &= \sqrt{1-(c\Delta t/\Delta x)^2}\,\Delta x \\ &= \sqrt{64}m \approx 8m\,. \end{aligned}$$

The separations in the y and z directions stay 0 because the two frames have no relative motion in those directions.

(c) What is the smallest spatial separation between the two events in any inertial frame?

Consider the frame (primed) in which the two events occur simultaneously, as found in part (a). In this frame,

$$\Delta x' = \Delta x_0 \, , \Delta t' = 0$$

where Δx_0 was determined in part (b). Now consider a Lorentz transformation to some other frame (double-primed):

$$\Delta x'' = \gamma(\Delta x' - \beta c \Delta t') = \gamma \Delta x_0$$

$$\Rightarrow |\Delta x''| > |\Delta x_0|$$

So, the ''proper length'' of a pair of events is the smallest spatial seperation observed.

In our case, $\beta = 0.6$ and the minimum spatial separation is $8m$ as calculated in parts (a) and (b).

6. Events in two different frames II (7 points)

Two events, A and B, are observed in two different inertial frame, Σ, and Σ'. Event A occurs at the spacetime origin in both frames,

$$x_A = y_A = z_A = ct_A = 0$$
$$x'_A = y'_A = z'_A = ct'_A = 0$$

Event B occurs at

$$x_B = 2, \quad y_B = z_B = 0, \quad ct_B = 10$$

(all distances are in meters) as observed in Σ.

The two events occur *at the same point* in frame Σ'.

(a) Find the velocity of Σ' with respect to Σ.

$$\begin{aligned} 0 &= \Delta x' \\ \Rightarrow 0 &= \gamma\,(\Delta x - \beta c \Delta t) \\ \Rightarrow \beta &= \frac{\Delta x}{c \Delta t} \\ \Rightarrow v &= 0.2c\,. \end{aligned}$$

Note that $|c\Delta t|$ has to be larger than $|\Delta x|$ so that $|\beta| < 1$. Such events are called time-like separated and there always exists a frame (Σ' in this case) in which two time-like separated events occur at the same point.

(b) What is the time separation of the two events in the frame Σ'?

$$\begin{aligned} c\Delta t' &= \gamma\,(c\Delta t - \beta\Delta x) \\ &= \sqrt{1-(\Delta x/(c\Delta t))^2}\,c\Delta t \\ &= \sqrt{96} = 9.8m\,. \end{aligned}$$

(c) What is the shortest time separation between the two events in any inertial frame?

`Consider the frame (primed) in which the two events occur at the same point, as found in part (a). In this frame,`

$$\Delta x' = 0\,, \Delta t' = \Delta t_0$$

`where` Δt_0 `was determined in part (b). Now consider a Lorentz transformation to some other frame (double-primed):`

$$\Delta t'' = \gamma(\Delta t' - \beta\Delta x'/c) = \gamma\Delta t_0$$

$$\Rightarrow |\Delta t''| > |\Delta t_0|$$

`So, the ''proper time separation'' of a pair of events is the smallest time separation observed.`
`In our case,` $\beta = 0.2$ `and the minimum temporal separation is` $\frac{9.8m}{c}$ `as calculated in parts (a) and (b).`

7. Events in two different frames III (7 points)

Two events, A and B, are observed in two different inertial frame, Σ, and Σ'. Event A occurs at the spacetime origin in both frames,

$$\begin{aligned} x_A = y_A = z_A = ct_A = 0 \\ x'_A = y'_A = z'_A = ct'_A = 0 \end{aligned}$$

Event B occurs at

$$x_B = 2, \quad y_B = z_B = 0, \quad ct_B = 3$$

in Σ, and at

$$x'_B = 3, \quad y'_B = z'_B = 0$$

(all distances are in meters) in Σ'.

(a) What time does event B occur in Σ'?

`The spacetime interval is a Lorentz invariant quantity. So,`

$$(ct_B)^2 - x_B^2 - y_B^2 - z_B^2 = (ct'_B)^2 - (x'_B)^2 - (y'_B)^2 - (z'_B)^2$$

$$\Rightarrow ct'_B = \pm\sqrt{(ct_B)^2 - (x_B^2 - (x'_B)^2) - (y_B^2 - (y'_B)^2) - (z_B^2 - (z'_B)^2)}$$

$$\Rightarrow ct'_B = \pm\sqrt{3^2 - (2^2 - 3^2)}m = \pm\sqrt{14}m \approx \pm 3.74m$$

To determine the correct sign, we use the fact that B is a time-like event and so the sign of the time coordinate is a Lorentz-invariant. To convince yourself of this, just think of the spacetime diagram and the fact that the Lorentz transformed spatial axis always lies between $ct = -\beta x$ and $ct = \beta x$.
Hence, $ct'_B \approx 3.74m$.

(b) What is the relative velocity of Σ' relative to Σ?

$$\begin{aligned} x'_B &= \gamma(x_B - \beta ct_B) \\ \Rightarrow \quad 3 &= \frac{2 - 3\beta}{\sqrt{1-\beta^2}} \end{aligned}$$

Squaring gives

$$18\beta^2 - 12\beta - 5 = 0$$

$$\Rightarrow \quad \beta = \frac{12 \pm \sqrt{504}}{36}$$

$$\Rightarrow \beta \approx 0.957 \text{ or } -0.290$$

Which is it? $\beta = 0.957$ gives $x'_B = -3m$ whereas $\beta = -0.290$ gives the correct $\beta = 3m$.
Hence, $\beta \approx -0.290$ which corresponds to a speed of $0.870 \times 10^8 m/s$ in the negative x direction.

8. An alternative derivation of the velocity transformation law (9 points)

Spaceship A passes earth at speed v when its clock and the adjacent earth clock read zero (event E_1). When the earth clock reads T, spaceship B passes earth, moving at speed $u > v$ in the same direction as spaceship A (event E_2). Eventually ship B catches up with ship A (event E_3). Let S be the earth frame and S' be the frame of ship A.

(a) Is the interval between E_1 and E_2 a proper time interval in either frame S or frame S'? If so, which one? What about the interval between E_2 and E_3? What about the interval between E_1 and E_3?

The events E_1 and E_2 happen at the same place (Earth) in frame S, so the interval between them is a proper time interval in S. The events E_2 and E_3 happen at different places in both S and S', so the interval is not a proper time interval in either frame (it is in the rest from of spaceship B, however). The evens E_1 and E_3 both happen at the same place (position of spaceship A) in frame S' so the tim einterval is a proper time interval in S'.

(b) Find the time of E_2 according to the S' clocks.

`Lorentz transformation between` S `and` S' `frame:`

$$\Delta T'_{12} = \gamma_v(\Delta T_{12} - v/c^2 \Delta x_{12}) = \gamma_v \Delta T_{12} = \gamma_v T$$

`Where`

$$\gamma_v \equiv \frac{1}{\sqrt{1 - v^2/c^2}}$$

(c) According to S' observers, how far away was the earth at E_2?

`In frame` S'`, the earth moves with velocity` v`, so its distance from spaceship A at` E_2 `is`

$$\Delta x'_{12} = v \Delta T'_{12} = v \gamma_v T$$

(d) Find the time of E_3 according to S' clocks.

`If the velocity of spaceship B as measured in` S' `is` v'`, the time it takes for spaceship B to meet A, i.e. travel a distance of` d'_2`, is`

$$\Delta T'_{23} = \frac{d'2}{v'} = \frac{v \gamma_v T}{v'}$$

`So the time of` E_3 `acording to` S' `clocks is`

$$T'_3 = T'_2 + \Delta T'_{23} = \gamma_v T (1 + \frac{v}{v'})$$

(e) Find the time of E_3 according to S clocks.

`In frame` S`, if the time elapsed between` E_2 `and` E_3 `is` ΔT_{23}`, we have`

$$v(\Delta T_{23} + T) = u \Delta T_{23}$$

`since spaceship A is` $v\Delta T$ `away from the Earth at` E_2`. Hence`

$$\Delta T_{23} = \frac{v}{u - v} T$$

`So the time of` E_3 `in frame` S `is:`

$$T_3 = T + \Delta T_{23} = \frac{u}{u - v} T$$

(f) From these results find the velocity of ship B as measured by observers on ship A (ie, S' observers). This is the relativistic velocity transformation law derived in lecture from the Lorentz transformation.

Since E_1 and E_3 both happen at the same place in S', we must have

$$T_3 = \gamma_v T_3'$$

So

$$\frac{uT}{u-v} = \gamma_v \left[\gamma_v T(1+\frac{v}{v'})\right]$$

$$1+\frac{v}{v'} = \frac{u(1-v^2/c^2)}{u-v}$$

$$\frac{v}{v'} = \frac{u - uv^2/c^2 - u + v}{u-v}$$

$$\Rightarrow v' = v\frac{u-v}{v-uv^2/c^2} = \frac{u-v}{1-uv/c^2}]$$

This is what we whould have gotten if we used the velocity transformation law to find the velocit of spaceship B as measured from the rest frame of spaceship A.

9. The expanding universe (RH) (7 points)

(a) Galaxy A is reported to be receding from us with a speed of $0.3c$. Galaxy B, located in precisely the opposite direction, is also found to be receding from us at this same speed. What recessional speed would an observer on Galaxy A find i) for our galaxy? ii) for Galaxy B?

(i) Since in our galaxy we observe Galaxy A to recede with a speed of $0.3c$, an observer in Galaxy A would observe our galaxy receding at the same speed $0.3c$. This is the relative speed between the two frames.
(ii) Let our galaxy be the unprimed frame and Galaxy A be the primed frame. Let the x-axis be directed from our galaxy to Galaxy A. In our frame, the velocity of Galaxy B is

$$u_x = -0.3c, u_y = u_z = 0\,.$$

Then, the velocity of Galaxy B as seen from Galaxy A is

$$u_x' = \frac{u_x - v}{1-u_x v/c^2} = \frac{-0.3c - 0.3c}{1-(-0.3\times 0.3)} \approx -0.55c$$

$$u_y' \propto u_y = 0\,, u_z' \propto u_z = 0\,.$$

So, an observer on Galaxy A sees Galaxy B receding with a speed of $0.55c$.

(b) It is concluded from measurements of the red shift of the emitted light that quasar Q_1 is moving away from us at a speed of $0.7c$. Quasar Q_2, which lies in the same direction in space, but is closer to us, is moving away from us at speed $0.55c$. What velocity for Q_2 would be measured by an observer on Q_1?

Let the x-axis be directed from us to Q_1.
Q_1 moves with a velocity $v = 0.7c$ relative to us.
The velocity of Q_2 in our frame is

$$u_x = 0.55c\,, u_y = u_z = 0\,.$$

Then, the velocity of Q_2 in the frame of Q_1 is

$$u'_x = \frac{u_x - v}{1 - u_x v/c^2} = \frac{0.55c - 0.7c}{1 - 0.55 \times 0.7} \approx -0.24c\,,$$

$$u'_y = 0\,, u'_z = 0\,.$$

So, an observer on Q_1 sees Q_2 receding at a speed of $0.24c$.

10. Perpendicular velocities (RH) (7 points)

The velocity of train B with respect to the station is

$$u_{B,E} = 0.75c\,, u_{B,N} = 0\,,$$

where the second subscript refers to the direction (E for East and N for North). The velocity of train A with respect to the station is

$$u_{A,E} = 0\,, u_{A,N} = 0.75c\,.$$

(a) Find $\vec{V}_{AB}$, the velocity of train B with respect to train A.

The velocity of train B relative to train A is

$$u_{AB,N} = \frac{u_{B,N} - u_{A,N}}{1 - u_{B,N}u_{A,N}/c^2} = \frac{0 - 0.75c}{1 - 0} = -0.75c\,,$$

$$u_{AB,E} = u_{B,E}\frac{\sqrt{1 - u_{A,N}^2/c^2}}{1 - u_{B,N}u_{A,N}/c^2} = 0.75c\frac{\sqrt{1 - 0.75^2}}{1} \approx 0.50c\,.$$

So, $\vec{u}_{AB}$ is $0.90c$ directed about $56°$ South of East.

(b) Find $\vec{V}_{BA}$, the velocity of train A with respect to train B.

The velocity of train A relative to train B is

$$u_{BA,E} = \frac{u_{A,E} - u_{B,E}}{1 - u_{A,E}u_{B,E}/c^2} = \frac{0 - 0.75c}{1 - 0} = -0.75c\,,$$

$$u_{BA,N} = u_{A,N}\frac{\sqrt{1 - u_{B,E}^2/c^2}}{1 - u_{A,E}u_{B,E}/c^2} = 0.75c\frac{\sqrt{1 - 0.75^2}}{1} = 0.50c\,.$$

So, $\vec{u}_{BA}$ is $0.90c$ directed about 56° West of North.

(c) Comment on the fact that these two relative velocities do not point in opposite directions.
(Ignore the font problem with East and the velocity labels.)

The angle between $\vec{u}_{AB}$ and $\vec{u}_{BA}$ is about $(2{\times}59^\circ)$ or 118°. The problem is symmetric with respect to projection into the 45○ line passing through origin. (The vector that is pointing East will transform to the vector that is pointing North and vice versa) and you see that the the vectors you obtained pass this symmetry test.

NOTE: Your result doesn't contradict the principle of relativity since there are actually 3 frames involved in this problem: frame S, frame attached to A, frame attached to B. So it seems that in going from train A's frame to train B's frame (by two successive perpendicular Lorentz transformations), our coordinate axes have been rotated.

11. Transforming angles I (7 points)

A particle moves with speed u in the $x - y$ plane, making an angle θ with respect to the x-axis in frame Σ. The origin of Σ moves to the right (along the positive x' axis) in the frame Σ' with speed v. What speed u' and angle θ' will the particle appear to have to an observer in Σ'?

Σ' has a speed v relative to Σ in the negative x direction.
The velocity of the particle in frame Σ is

$$u_x = u\cos\theta\,, u_y = u\sin\theta\,.$$

Therefore the velocity in frame Σ' is

$$u'_x = \frac{u_x - (-v)}{1 - u_x(-v)/c^2} = \frac{u\cos\theta + v}{1 + uv\cos\theta/c^2}\,,$$

$$u'_y = u_y\frac{\sqrt{1 - v^2/c^2}}{1 - u_x(-v)/c^2} = \frac{u\sin\theta}{\gamma(1 + uv\cos\theta/c^2)}\quad.$$

So,

$$u' = \sqrt{(u'_x)^2 + (u'_y)^2} = \frac{\sqrt{v^2 + u^2(1-\beta^2\sin^2\theta) + 2uv\cos\theta}}{1 + uv\cos\theta/c^2},$$

$$\theta' = \tan^{-1}\frac{u'_y}{u'_x} = \tan^{-1}\frac{u\sin\theta}{\gamma(u\cos\theta + v)}.$$

12. Transforming angles II (7 points)

A right triangular plate is at rest in the frame Σ. Its legs are placed on the x and y axes and its hypotenuse makes an angle θ with respect to the x-axis. The origin of Σ moves to the right (along the positive x' axis) in the frame Σ' with speed v. What are the angles of the triangle as measured in Σ'?

Σ' has a speed v relative to Σ in the negative x direction.
In frame Σ, the triangle has a length l_x along the x-axis and l_y along the y-axis with

$$tan\theta = \frac{l_y}{l_x}.$$

In frame Σ', the legs of the triangle have lengths

$$l'_x = \frac{l_x}{\gamma}, l'_y = l_y.$$

So, the angle that the hypotenuse makes with the x'-axis is

$$\theta' = \tan^{-1}\frac{l'_y}{l'_x} = \tan^{-1}(\gamma\tan\theta).$$

13. Doppler shift (7 points)

Resnick Chapter 2, Problem 48, P 108.

Source and observer are approaching each other therefore wavelength will be blue shifted and they are given by

$$\lambda = \sqrt{\frac{1-\beta}{1+\beta}}\ \lambda_0,$$

where $\lambda_0 = 5896\,A^\circ$.

For $\beta = 0.1$

$$\lambda = 5333A^\circ.$$

For $\beta = 0.4$

$$\lambda = 3860A^\circ.$$

For $\beta = 0.8$

$$\lambda = 1965A^{\circ}.$$

Classical (first order) result is not a good approximation.

14. Won't this excuse get you in worse trouble? (7 points)

The wavelength of red light is 650 nm and the wavelength of yellow light is 570 nm. You run a red light, and a policeman pulls you over. You tell him that because of the Doppler shift, you thought the light was still yellow. If the policeman believes you and has taken 8.20, how fast does he conclude you were driving?

The traffic light emits red light with wavelength $\lambda_0 = 650$ nm.
You approach the light with a speed parameter β.
You claim that the light was Doppler shifted to yellow with wavelength $\lambda = 570$ nm.
Here is what the policeman concludes:

$$\begin{aligned} \lambda &= \sqrt{\frac{1-\beta}{1+\beta}}\,\lambda_0 \\ \Rightarrow \beta &= \frac{\lambda_0^2 - \lambda^2}{\lambda_0^2 + \lambda^2} \\ &= \frac{976}{7474} \approx 0.13 \\ \Rightarrow v &\approx 0.39 \times 10^8 m/s \text{ or } 8.82 \times 10^7\, miles/hr \ ! \end{aligned}$$