Physics 8.20 Special Relativity IAP 2008

Problem Set #3 Solutions

1. **Longitudinal and transverse Doppler shifts in terms of wavelengths** (5 points)

   (a) Show that the Doppler shift formulas can be written in the following forms:

$$
\begin{aligned}
\lambda &= \lambda_0(1-\beta+\frac{1}{2}\beta^2+\ldots) \quad \text{approaching} \\
\lambda &= \lambda_0(1+\beta+\frac{1}{2}\beta^2+\ldots) \quad \text{receeding} \\
\lambda &= \lambda_0(1+\frac{1}{2}\beta^2+\frac{3}{8}\beta^4+\ldots) \quad \text{transverse}
\end{aligned}
$$

   where λ_0 is the wavelength measured by an observer at rest with respect to the source.

   When the source and the observer approach each other, then the observed wavelength, λ, is blue shifted according to

$$
\begin{aligned}
\lambda &= \sqrt{\frac{1-\beta}{1+\beta}}\lambda_0 \\
&= \lambda_0(1-\frac{1}{2}\beta-\frac{1}{8}\beta^2+\ldots)(1-\frac{1}{2}\beta+\frac{3}{8}\beta^2+\ldots) \\
&= \lambda_0(1-\beta+\frac{1}{2}\beta^2+\ldots)\,.
\end{aligned}
$$

   When the source recedes from the observer, the observed wavelength is redshifted and is obtained from the above formula with $\beta \to -\beta$:

$$
\begin{aligned}
\lambda &= \sqrt{\frac{1+\beta}{1-\beta}}\lambda_0 \\
&= \lambda_0(1+\beta+\frac{1}{2}\beta^2+\ldots)\,.
\end{aligned}
$$

   When the source moves transverse to the observer, then the observed wavelength is redshifted (a purely relativistic effect) according to

$$
\begin{aligned}
\lambda &= \gamma\lambda_0 \\
&= \lambda_0(1+\frac{1}{2}\beta^2+\frac{3}{8}\beta^4+\ldots)\,.
\end{aligned}
$$

   (b) Derive a similar expression valid through order β^2 when the source makes an angle θ relative to the direction away from the observer.

When the source velocity makes an angle θ with the direction away from the observer,

$$\lambda = \lambda_0 \gamma (1 + \beta \cos\theta)\,.$$

This of course reduces to the receding, approaching and transverse expressions for $\theta = 0, \pi$ and $\pi/2$ respectively.
Expanding to order β^2,

$$\lambda = \lambda_0 (1 + \beta \cos\theta)(1 + \frac{1}{2}\beta^2 + \ldots) = \lambda_0 (1 + \beta \cos\theta + \frac{1}{2}\beta^2 + \ldots)\,.$$

2. **The Most Distant Galaxy Known:** (3 points)

One of the most distant galaxies that has been observed is described as "having a redshift of $z = 6.58$". In this problem, we explore what this means. This galaxy was discovered in 2003 by the Subaru Telescope, a facility supported by the National Astronomical Observatory of Japan. You can read about it at the following link: <http://www.naoj.org/Pressrelease/2003/03/>.

Hot hydrogen gas emits light with a particular set of frequencies, referred to as emission lines. The spectrum of emission lines serves as a finger print, allowing an astronomer who observes it to deduce that she is seeing hot hydrogen gas. The "Lyman-α line" is a prominent feature of the hydrogen spectrum which has a wavelength of 121.6 nm, or 1.216×10^{-7} m. This is the wavelength measured in a laboratory experiment, in which both the hot hydrogen and the detection apparatus (the "telescope") are at rest. [Note that this wavelength is deep in the ultraviolet. The human eye is sensitive to light with wavelengths from about 400 nm (violet light) to about 700 nm (red light).]

The Subaru telescope group found a galaxy in which the Lyman-α line has wavelength 922 nm. The Doppler shift is so great that ultraviolet light has become infrared! By convention, the redshift z is defined as $1 + z =$(observed wavelength/emitted wavelength). In this case, $1 + z = 922/121.6$, yielding $z = 6.58$.

Use the formula for the special relativistic Doppler shift to deduce the relative velocity between the earth and the distant galaxy, assuming that the galaxy is receding directly away from the earth. (This is a good approximation; any transverse velocity would be much smaller than the recession velocity you have calculated.)

Suppose that the quasar is receding with a speed parameter β.
It emits the Lyman-α line with a wavelength $\lambda_0 = 121.6nm$.
This is observed as $\lambda = 885 \pm 3nm$.

$$\begin{aligned}
\lambda &= \sqrt{\frac{1+\beta}{1-\beta}}\lambda_0 \\
\Rightarrow \beta &= \frac{\lambda^2 - \lambda_0^2}{\lambda^2 + \lambda_0^2} \\
&\approx 0.966
\end{aligned}$$

3. Visual appearance of a rod (7 points)

This problem essentially recapitulates the discussion in lecture.

Suppose a rod of rest length ℓ_0 is oriented along the x' axis in the frame S'. The frame S' moves to the right with velocity v along the x-axis when viewed from the frame S. The origins of S and S' coincide at $t = t' = 0$ and their axes are parallel.

An observer sits at the origin of S and watches the rod by looking at the light omitted by it.

(a) What is the *apparent* length of the rod at for times > 0?

We will work in the frame Σ. The measured length of the rod is length contracted to

$$\frac{\ell_0}{\gamma}.$$

Choose coordinates so that at $t = 0$, the left end of the rod is at $x = 0$ (where the observer is) and the right end is at $x = \ell_0/\gamma$.
Event 1: Light emitted from the right end of the rod at coordinates

$$t_R\,, x_R = \ell_0/\gamma + vt_R\,.$$

Event 2: Light emitted from the left end of the rod at coordinates

$$t_L\,, x_L = vt_L\,.$$

If these two light signals reach the observer simultaneously (say at $t = t_O$), then

$$c(t_O - t_R) = x_R = \frac{\ell_0}{\gamma} + vt_R\,,$$
$$c(t_O - t_L) = x_L = vt_L$$
$$\Rightarrow t_L - t_R = \frac{\ell_0}{c\gamma(1+\beta)}$$

The apparent (or seen) length of the rod is

$$\begin{aligned} x_R - x_L &= \frac{\ell_0}{\gamma} - v(t_L - t_R) \\ &= \frac{\ell_0}{\gamma(1+\beta)} \\ &= \ell_0\sqrt{\frac{1-\beta}{1+\beta}}\,. \end{aligned}$$

Note that for $\beta > 0$ (i.e. a receding rod), the apparent length is shorter than the measured contracted length ℓ_0/γ. And for $\beta < 0$ (i.e. an approaching rod), the apparent length is longer than the proper length ℓ_0.

(b) How do you reconcile this result with Lorentz Contraction?

This result does not contradict Lorentz contraction.
The measured length of a rod is the spatial separation between its two ends at a fixed time t. This length is contracted by the Lorentz factor in the direction of relative motion between the rod and the observer.
The apparent length of the rod, on the other hand, is the spatial separation between its two ends measured at two different times (t_1 and t_2) with the two times related in such a way that light emitted from one end at $t = t_1$ reaches the observer at the same time as the light emitted from the other end at $t = t_2$.

(c) Repeat (a) and (b) for velocity $-v$.

It's discussed above.

4. Lorentz Transformation of Solutions to Maxell's Equations (6 points)

You have previously shown that Maxwell's wave equation

$$\frac{\partial^2}{\partial x^2}E(x,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}E(x,t) = 0 \tag{1}$$

is invariant under Lorentz transformations and that functions of the form $E(kx - kct)$ satisfy it and correspond to motion to the right with velocity c. Here k is a constant inserted for later convenience. Consider the solution

$$E(x,t) = \sin(kx - kct),$$

which describes a sinusoidal wave with wavelength $\lambda = 2\pi/k$, periond $T = 2\pi/kc$, and frequency $\nu = kc/2\pi$

(a) By direct transformation of $(x - ct)$ to a frame S' moving with speed v relative to S, obtain $E(x', t')$. Show your result for $E(x', t')$ satisfies the wave equation in the S' frame:

$$\frac{\partial^2}{\partial x'^2}E(x',t') - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}E(x',t') = 0$$

Write the Lorentz transformations:

$$x = \gamma(x' + \beta\ ct')$$

$$ct = \gamma(ct' + \beta\ x')$$

$$\Rightarrow x - ct = \gamma(1-\beta)(x' - ct')$$

$$E(x',t') = \sin\big(k\gamma(1-\beta)(x' - ct')\big) \tag{2}$$

$$\frac{\partial^2}{\partial x'^2}E(x',t') = -k^2\gamma^2(1-\beta)^2 \sin\big(k\gamma(1-\beta)(x'-ct')\big)$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}E(x',t') = -\frac{k^2\gamma^2(1-\beta)^2c^2}{c^2}\sin\big(k\gamma(1-\beta)(x'-ct')\big) =$$

$$-k^2\gamma^2(1-\beta)^2 \sin\big(k\gamma(1-\beta)(x'-ct')\big) = \frac{\partial^2}{\partial x'^2}E(x',t')$$

(b) The solution in the S' frame, $E(x',t')$, should again be a sinusoidal wave traveling to the right. Determine its wavelength λ', period T', and frequency ν'. Check that $\lambda'\nu'$ yields the correct speed of light. Check that your formulae agree with the Doppler shift.

`>From equation (2) you can read` k':

$$k' = k\gamma(1-\beta)$$

$$\omega' = k\gamma(1-\beta)c$$

`>From the relations:`

$$\lambda' = \frac{2\pi}{k'}$$

$$T' = \frac{2\pi}{\omega'}$$

`we get`

$$\lambda' = \frac{\lambda}{\gamma(1-\beta)} = \lambda\sqrt{\frac{1+\beta}{1-\beta}}$$

$$T' = \frac{T}{\gamma(1-\beta)} \Rightarrow \nu' = \frac{1}{T'} = \nu\gamma(1-\beta) = \nu\sqrt{\frac{1-\beta}{1+\beta}}$$

`And the relation:`

$$\nu'\lambda' = \nu\lambda = c$$

`is easily checked.`

5. Sequential Lorentz Transformations French, problem 5-7, p 160. (6 points)

`Write the Lorentz transformations in the matrix form:`

$$\begin{pmatrix} x_1 \\ ct_1 \end{pmatrix} = \begin{pmatrix} \gamma_1 & -\gamma_1\beta_1 \\ -\gamma_1\beta_1 & \gamma_1 \end{pmatrix}\begin{pmatrix} x \\ ct \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ ct_2 \end{pmatrix} = \begin{pmatrix} \gamma_2 & -\gamma_2\beta_2 \\ -\gamma_2\beta_2 & \gamma_2 \end{pmatrix}\begin{pmatrix} x_1 \\ ct_1 \end{pmatrix}$$

Combine the above equations:

$$\begin{pmatrix} x_2 \\ ct_2 \end{pmatrix} = \begin{pmatrix} \gamma_2 & -\gamma_2\beta_2 \\ -\gamma_2\beta_2 & \gamma_2 \end{pmatrix} \begin{pmatrix} \gamma_1 & -\gamma_1\beta_1 \\ -\gamma_1\beta_1 & \gamma_1 \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ ct_2 \end{pmatrix} = \begin{pmatrix} \gamma_2\gamma_1(1+\beta_2\beta_1) & -\gamma_2\gamma_1(\beta_2+\beta_1) \\ -\gamma_2\gamma_1(\beta_2+\beta_1) & \gamma_2\gamma_1(1+\beta_2\beta_1) \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}$$

The next step is to write

$$\gamma_1\gamma_2(1+\beta_1\beta_2)$$

as some $\gamma(\beta)$:

$$\gamma_1\gamma_2(1+\beta_1\beta_2) = \frac{1}{\sqrt{1-\beta_1^2}}\frac{1}{\sqrt{1-\beta_2^2}}\sqrt{(1+\beta_1\beta_2)^2}$$

$$= \frac{1}{\sqrt{\frac{(1-\beta_1)^2(1-\beta_2^2)+(1+\beta_1\beta_2)^-(1+\beta_1\beta_2)^2}{(1+\beta_1\beta_2)^2}}} = \frac{1}{\sqrt{1-\left(\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}\right)^2}} = \gamma(\frac{\beta_1+\beta_2}{1+\beta_1\beta_2})$$

The second check is to find the ratio:

$$\frac{-\gamma_2\gamma_1(\beta_2+\beta_1)}{\gamma_2\gamma_1(1+\beta_2\beta_1)} = -\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}$$

This proves that:

$$\begin{pmatrix} x_2 \\ ct_2 \end{pmatrix} = \begin{pmatrix} \gamma(\beta) & -\beta\gamma(\beta) \\ -\beta\gamma(\beta) & \gamma(\beta) \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}$$

with

$$\beta = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}.$$

6. Lorentz transformations as rotations French, problem 3-9, p 87. (5 points)

You saw in class that

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Lorentz transformation takes the form

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix} \tag{3}$$

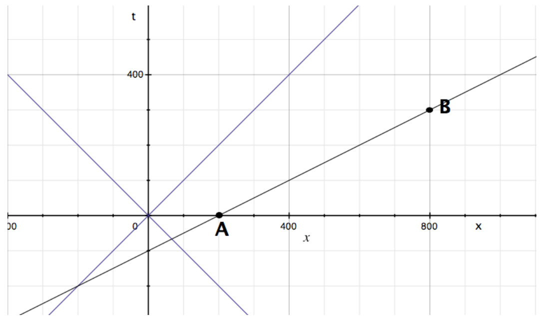

Figure 1: space-time graph for 12(a). Blue lines represent $x = \pm ct$. Length and time is measured in meters.

where

$$\tanh\phi \equiv \beta \ \cdot$$

Write equation (3) in for (x, ict):

$$\begin{pmatrix} x' \\ ict' \end{pmatrix} = \begin{pmatrix} \cosh\phi & +i\sinh\phi \\ -i\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} x \\ ict \end{pmatrix} \tag{4}$$

Now use the relations:

$$\cos i\phi = \cosh\phi \qquad \sin i\phi = i\sinh\phi \ .$$

to rewrite equation (4):

$$\begin{pmatrix} x' \\ ict' \end{pmatrix} = \begin{pmatrix} \cos i\phi & \sin i\phi \\ -\sin i\phi & \cos i\phi \end{pmatrix} \begin{pmatrix} x \\ ict \end{pmatrix}$$

7. **Space-time diagrams I** Resnick, Supplementary topic A problem 5, pp 200 (10 points)

Lets put $c = 1$ and measure time in units of length i.e. $1\,s = 3\times 10^8\,m$ or $10^{-6}\,s = 300\,m$.

12(a) Points A and B are plotted on the space-time graph (figure 1). If they are simultaneous in some frame then line joining A and B should be parallel to $t' = 0$ line or x' axis in that frame. This is possible only if slope of line AB less than 1. Comparing directly with $x = ct$ line we see that line AB has slope less than 1. Verify by evaluating space-time interval

$$c^2\tau^2 = c^2t^2 - x^2 = 300^2 - 600^2 = -27\times 10^4\,m^2$$

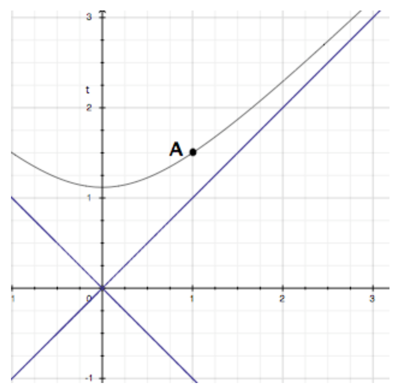

Figure 2: space-time graph for 13(a). Blue lines represent $x = \pm ct$. Length and time is measured in $10^9\, m$.

which is space-like interval hence we can make events simultaneous.

13(a) For convenience lets take one of the events at the origin. Second event is plotted at point A in figure 2 at $t = 5\,s = 15 \times 10^8\, m$. It is evident from the graph that events are time-like separated. We draw a hyperbola passing through point A which intersects with t-axis and the t intercept gives the invariant space-time interval. We measure it to be 1.2 on the graph, therefore $c\tau = 1.12 \times 10^9\, m$. Verify it directly by evaluating

$$c^2\tau^2 = c^2t^2 - x^2 = (15 \times 10^8)^2 - 10^18 = 1.25 \times 10^18\, m^2$$

whose square-root gives $1.118 \times 10^9\, m$.

8. **Space-time diagrams II** Resnick, Supplementary topic A problem 8, p 200 (8 points)

We wish to find velocity of S'' w.r.t. S'. The easiest way to do this is look at the worldline of a point in S'' which remains at rest w.r.t. S''. Obvious choice is origin whose worldline is t'' axis. Now choose any point on t'' axis and find its x' and t' co-ordinates. We choose point A on t'' line. Line AP gives t'-intercept and and AQ gives x'-intercept. Now velocity of a particle along the worldline t'' in S' is given by

$$V_{S''\texttt{w.r.t.}S'} = \frac{x' - 0}{t' - 0} = \frac{\frac{OQ}{OD}}{\frac{OP}{OC}} = \frac{OQ}{OP}.$$

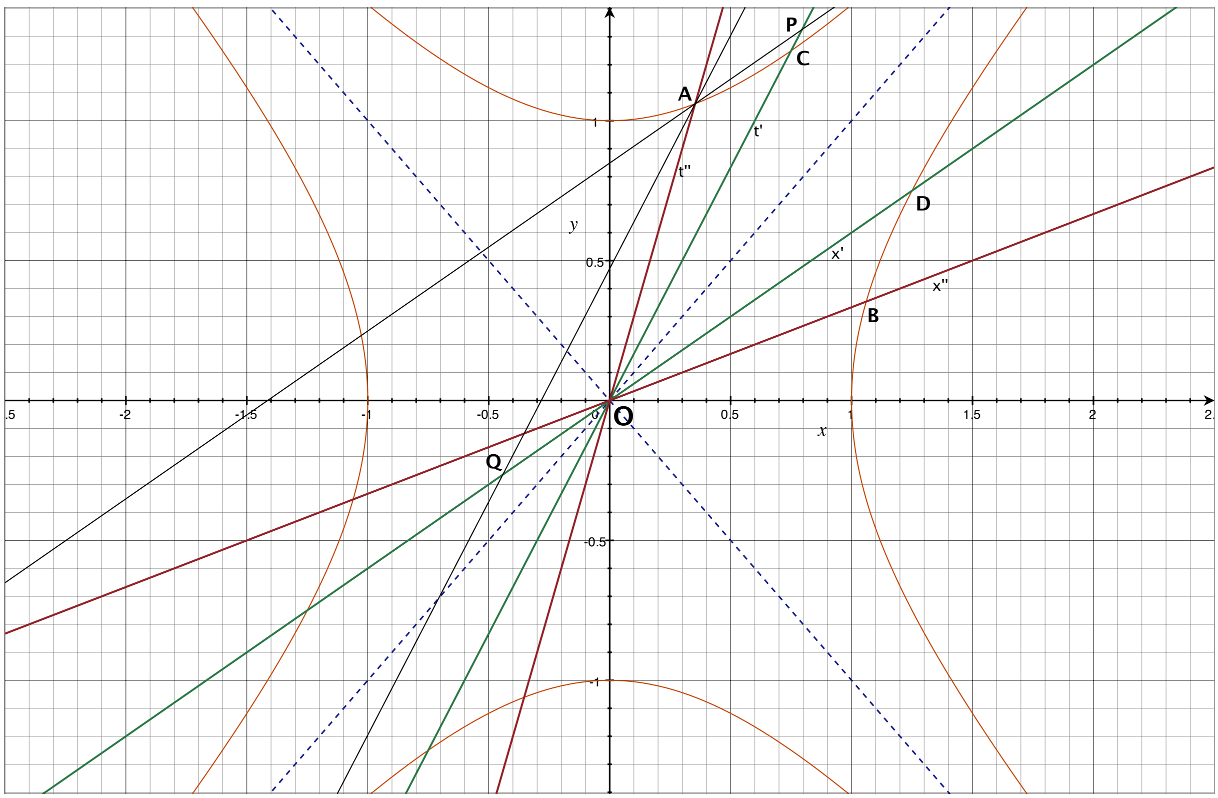

Figure 3: Space-time diagram for velocity addition: green lines represent $x' - t'$ axis of S' frame and red represent $x'' - t''$ axis of frame S''. Orange curves represent equations $c^2t^2 - x^2 = \pm 1$ and blue dashed lines represent the light cone. Points A, B, C and D are intersection of orange curves with space-time axes for frames S' and S''. Hence they give measurement of unit length and unit times in these frames. Line AP is parallel to x' axis and line AQ is parallel to t' axis. In this graph we have used units such that $c = 1$.

In previous equation we divide lengths OP and OQ by OD and OC because they are the unit lengths in S' frame but we observe that $OD = OC$ since we have set $c = 1$. Now all we have to do is measure lengths OP and OQ from the graph and evaluate above equation. We get $OQ = -0.515$ and $OP = 1.545$. This gives

$$V_{S''\texttt{w.r.t.}S'} = \frac{-0.515}{1.545} = -0.333 \texttt{ or } -\frac{1}{3}c$$

as expected from velocity addition.

9. **Another version of the famous polevaulter problem** (10 points)

The frames Σ' and Σ are related in the standard fashion. A (one dimensional) garage is at rest in Σ. Its front end is at $x = 0$; its rear end at $x = L_0$. The garage has doors at both ends. Initially the front door is open and the rear door closed.

A Stacy Dragila (a woman world champion pole vaulter) runs with velocity v up the x-axis as viewed from Σ. Of course, she is at rest in Σ'. She holds her pole, which is of rest length ℓ_0, parallel to the x-axis. [ℓ_0 is much greater than the rest length of the garage.] The right end of her pole reaches the left end of the garage at $t = t' = 0$. She runs so fast that Lorentz contraction shortens her pole to $\ell = L_0/2$ as viewed in Σ.

In the rest frame of the garage, it is clear that her pole fits into the garage, making the following sequence of events possible:

- **Event A** At the instant that the back end of the pole reaches the front of the garage, the front door of the garage is closed.
- **Event B** At the instant that the front end of the pole reaches the back end of the garage, the back door of the garage is opened.

Note that Event B occurs after Event A in Σ.

(a) What is Dragila's velocity in terms of L_0, ℓ_0 and c?

In the garage frame, the pole is length contracted to $L_0/2$. So,

$$\frac{\ell_0}{\gamma} = \frac{L_0}{2}$$

$$\Rightarrow \gamma = \frac{2\ell_0}{L_0}$$

$$\Rightarrow \beta = \sqrt{1 - \frac{1}{\gamma^2}} = \sqrt{1 - \frac{L_0^2}{4\ell_0^2}}$$

(b) Find the position and time of Events A and B in Σ.

Event A: Back end of the pole reaches the front end of the garage. The front of the garage is at

$$x_A = 0$$

for all t.
The back end of the pole is at $x = L_0/2$ at $t = 0$ and moves to the right with a speed parameter β calculated in part (a). So it reaches the front end of the garage (x=0) at a time

$$t_A = \frac{L_0}{2v} = \frac{\ell_0 L_0}{c\sqrt{4\ell_0^2 - L_0^2}}\,.$$

Event B: The front end of the pole reaches the back end of the garage. The back end of the garage is at

$$x_B = L_0$$

for all t.
The front end of the pole starts at $x = 0$ at $t = 0$ and reaches the back end of the garage at time

$$t_B = \frac{L_0}{v} = \frac{2\ell_0 L_0}{c\sqrt{4\ell_0^2 - L_0^2}}\,.$$

Note that $t_B > t_A$. In fact, $t_B = 2t_A$.

(c) Now consider the events in Stacy's rest frame. Transform Events A and B to Σ' and show that the back door opens before the front door closes, making it possible for Stacy and the pole to get in and out of the garage without being crushed.

The coordinates of event A in Σ' are:

$$\begin{aligned}
t'_A &= \gamma(t_A - \beta x_A/c) \\
&= \frac{2\ell_0}{L_0}\frac{\ell_0 L_0}{c\sqrt{4\ell_0^2 - L_0^2}} \\
&= \frac{2\ell_0^2}{c\sqrt{4\ell_0^2 - L_0^2}} \\
x'_A &= \gamma(x_A - \beta c t_A) \\
&= -\frac{2\ell_0}{L_0}\frac{\sqrt{4\ell_0^2 - L_0^2}}{2\ell_0}\frac{\ell_0 L_0}{\sqrt{4\ell_0^2 - L_0^2}} \\
&= -\ell_0
\end{aligned}$$

The coordinates of event B in Σ' are:

$$\begin{aligned}
t'_B &= \gamma(t_B - \beta x_B/c) \\
&= \frac{2\ell_0}{L_0}\left(\frac{2\ell_0 L_0}{c\sqrt{4\ell_0^2 - L_0^2}} - \frac{\sqrt{4\ell_0^2 - L_0^2}}{2\ell_0}\frac{L_0}{c}\right) \\
&= \frac{L_0^2}{c\sqrt{4\ell_0^2 - L_0^2}} \\
x'_B &= \gamma(x_B - \beta c t_B) \\
&= \frac{2\ell_0}{L_0}\left(L_0 - \frac{\sqrt{4\ell_0^2 - L_0^2}}{2\ell_0}\frac{2\ell_0 L_0}{\sqrt{4\ell_0^2 - L_0^2}}\right) \\
&= 0
\end{aligned}$$

The above results could have been obtained (perhaps more easily) by working directly in the pole frame. Note that $t'_B < t'_A$ for $\ell_0 > L_0/\sqrt{2}$. Hence, the back door of the garage is opened before the front door closes making it possible for Stacy and the pole to get in and out of the garage without being crushed.
An aside: You may wonder that $t'_B < t'_A$ only when $\ell_0 > L_0/\sqrt{2}$. And from part part(a) we know that $\ell_0 > L_0/2$. So what happens when $L_0/2 < \ell_0 < L_0/\sqrt{2}$? In that case, the pole is shorter than the garage in either frame and there is no paradox to begin with!

(d) After getting bored opening and closing doors at pre-assigned times during a practice session, the door operators (in Σ) propose another scheme: The front door closer proposes to send a signal to the back door opener as soon as he closes the front door (Event A) telling him to open the back door. Will this signal reach the back door in time to be effective?

The front door closer can send a signal to the back door closer with a maximum possible speed of c. In the garage frame (Σ), this signal will reach the back door at time

$$\begin{aligned}
t_{B,sig} &= t_A + \frac{L_0}{c} \\
&= \frac{\ell_0 L_0}{c\sqrt{4\ell_0^2 - L_0^2}} + \frac{L_0}{c} \\
&= t_B\left(\frac{1}{2} + \sqrt{1 - \frac{L_0^2}{4\ell_0^2}}\right) \\
&> t_B
\end{aligned}$$

for $\ell_0 > L_0/\sqrt{2}$ (i.e. when the pole is longer than the garage in the rest frame of the pole). So, the signal will reach the back door opener only after the front end of the pole reaches her and will be ineffective.

10. The airplane falling through the ice (6 points)

In lecture, it was claimed that a rod (an "airplane") drifting down upon an ice sheet with a hole in it, sees the plane of the ice rotated in its rest frame. The aim of this problem is to establish this effect. Note that this is a conceptually challenging problem.

(a) First, let Σ and Σ' be frames related in the standard fashion with relative speed v. Suppose a rod at rest in Σ' and of rest length ℓ_0 makes an angle θ' with the x' axis. Find the length of the rod (ℓ) and the angle (θ) that it makes with the x axis as observed in Σ.

In the Σ frame, the rod will be contracted in the x-direction and will be undeformed in the y-direction. So,

$$\ell_x = \frac{\ell_0 \cos\theta'}{\gamma}\,, \ell_y = \ell_0 \sin\theta'$$

$$\Rightarrow \ell = \sqrt{\ell_x^2 + \ell_y^2} = \ell_0\sqrt{1-\beta^2\cos^2\theta'}$$

$$\text{And,}\quad \theta = \tan^{-1}\frac{\ell_y}{\ell_x} = \tan^{-1}(\gamma\tan\theta')$$

(b) Now introduce new (rotated) Cartesian coordinate axes $\tilde{x}'$ and $\tilde{y}'$ in Σ' so that the rod lies along the $\tilde{x}'$ axis. Let the $\tilde{x}$ and $\tilde{y}$ axes be defined to be parallel to the $\tilde{x}'$ and $\tilde{y}'$ axes when the origins of Σ and Σ' coincide. Show that the origin of Σ' has velocity components $(v_{\tilde{x}}, v_{\tilde{y}}) = (v\cos\theta', -v\sin\theta')$ viewed from Σ.

The $\tilde{x}, \tilde{y}$ axes are obtained by a counterclockwise rotation by θ' of the x, y axes. So, the components of a vector in the rotated basis will be obtained by a clockwise rotation by θ' of the components in the original (unrotated) basis.
The velocity of the origin of Σ' measured in the x, y basis of Σ is

$$v_x = v\,, v_y = 0\,.$$

So, the velocity in the rotated basis is

$$v_{\tilde{x}} = v_x\cos\theta' + v_y\sin\theta' = v\cos\theta'$$

$$\text{and,}\quad v_{\tilde{y}} = -v_x\sin\theta' + v_y\cos\theta' = -v\sin\theta'\,.$$

(c) Show that the rod, which lies along the $\tilde{x}'$ coordinate axis in Σ', is observed at an angle to the $\tilde{x}$ coordinate axis in Σ. What is the angle?

The rod makes an angle

$$\theta_x = \tan^{-1}(\gamma\tan\theta')$$

with the x axis. The $\tilde{x}$ axis makes an angle θ' with the x axis. Therefore, the rod makes an angle

$$\theta_{\tilde{x}} = \theta_x - \theta' = \tan^{-1}(\gamma \tan \theta') - \theta'$$

with the $\tilde{x}$ axis. This result may also be obtained (more tediously) by Lorentz transforming the coordinates of the rod in the $\tilde{x}', \tilde{y}'$ frame to the $\tilde{x}, \tilde{y}$ frame.

11. A Twin Problem French §5, Problem 5-19. (6 points)

(a) The numbers of years that go by in B frame are 2×4(lt yr)/(.8c) = 10 yrs. Hence B sends 10 signals. In the rest frame of A, alpha centuri moves towards him and earth moves away with a velocity of .8c, and then vice versa. But the distance to it is $\gamma \times 4$ lt yrs, which gives the total time for the trip in his frame as 6 yrs. So he can send only 6 signals.

(b) A receives only one siganl on his way to alpha centauri, and receives 9 on the way back, as can be seen in Fig.11b.

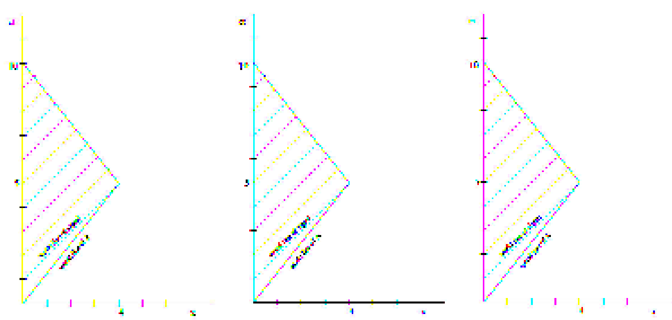

Figure 4: Problem 3 (b)

(c) B receives 3 signals during the first nine years and then receives 3 in the last year, as can be seen from Fig.11c.

(d) As A moves away from B, their siganls to each other are redshifted, as consecutive wave crests take longer and longer to reach the observer. On the way back, these signals now appear blue shifted to the respective observers. The redshift and blueshift is the same for both observers. Hence for redshift

$$\frac{\nu}{\nu_0} = \sqrt{\frac{1-\beta}{1+\beta}} = \frac{1}{3}.$$

The blueshift is

$$\frac{\nu}{\nu_0} = \sqrt{\frac{1+\beta}{1-\beta}} = 3.$$

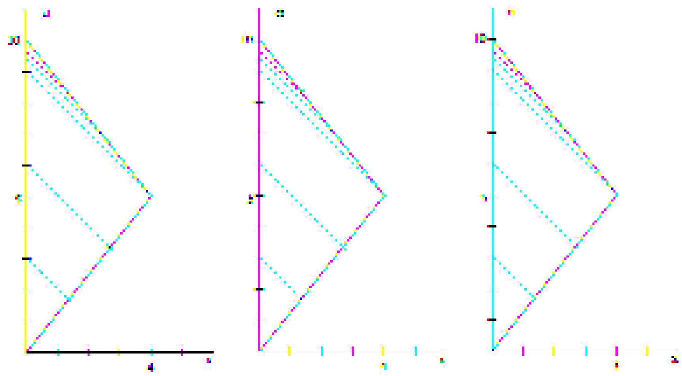

Figure 5: Problem 3 (c)

12. Einstein's "clock paradox" (4 points)

[Adapted from R. Resnick *Introduction to Special Relativity*]

Einstein, in his first paper on the special theory of relativity, wrote the following:

> "If one of two synchronous clocks at A is moved in a closed curve with constant velocity until it returns to A, the journey lasting t seconds, then by the clock that has remained at rest the travelled clock on its arrival at A will be $tv^2/2c^2$ seconds slow."

(a) Prove this statement. Note: elsewhere in his paper Einstein stated that this result is an approximation valid for $v \ll c$.

`>From the invariance of the spacetime interval,`

$$\begin{aligned} c\Delta\tau &= \sqrt{\Delta s^2} \\ &= \sqrt{(c\Delta t)^2 - (\Delta x)^2} \\ &= \sqrt{1 - (\frac{v}{c})^2} c\Delta t \\ \Rightarrow \Delta t - \Delta\tau &= (1 - \sqrt{1-\beta^2})\Delta t \\ &\approx \frac{1}{2}\beta^2 \Delta t. \end{aligned}$$

`Hence the moving clock is slower than the clock at rest by` $tv^2/2c^2$`, if the journey takes time` t `when` $v \ll c$`.`

(b) What is the exact statement?

`>From the calculation in the previous part, the exact statement contains` $(1 - \sqrt{1-\beta^2})t$ `instead of` $tv^2/2c^2$`.`

13. Twin Problems (4 points)

[Adapted from R. Resnick *Introduction to Special Relativity*]

This problem refers to Figure B-2 in Resnick. "Bob" is the travelling twin. "Dave" is the stay at home.

(a) In the spacetime diagram of Figure B-2 how far apart are Bob and Dave when Bob turns around?
`4 lt yrs.`

(b) Suppose that Dave did not know beforehand when Bob was planning to turn around. When (by his own clocks and calendars) would Dave know that Bob had done so?

`1 year before Bob's return, or 9 years after his departure.`

(c) Suppose that Bob, after noting the passage of three years by his on-board clock, decides not to return to Dave, but simply stops. He compares his on-board clock with one of the local clocks belonging to the synchronized array of stationary clocks fixed in Dave's inertial frame. What will his local clock read? Draw Bob's world line for this new situation on the diagram of Figure B-2.

`The local clock reads 5 lt yrs.  See Fig.13c.`

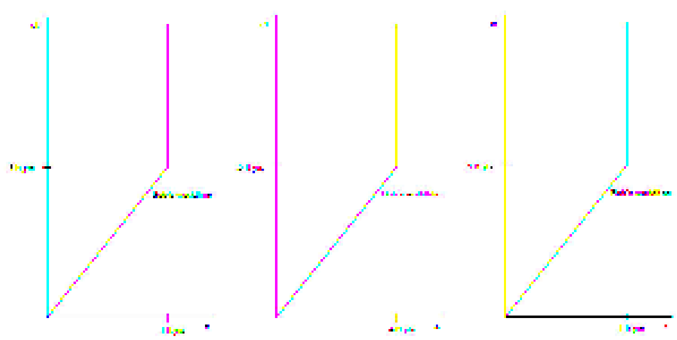

Figure 6: Problem 5 (c)

14. Bob is older than Dave this time! (10 points)

[From R. Resnick & D. Halliday *Basic Concepts in Relativity*]

Bob, once started on his outward journey from Dave, keeps on going at his original uniform speed of $0.8c$. Dave, knowing that Bob was planning to do this, decides, after waiting for three years, to catch up with Bob and to do so in just three additional years.

(a) To what speed must Dave accelerate to do this?

Call the initial rest frame of Dave Σ. Dave wants to reach Bob in his proper time $\tau = 3$ yrs. If Dave with velocity v_D takes time t in Σ to reach Bob who is moving with velocity $v_B = 0.8c$

$$\begin{aligned} v_B(t+3) &= v_D t \\ 3 = \tau_D &= \sqrt{1-\beta_D^2}\,t \\ \Rightarrow \frac{\tau}{\sqrt{1-\beta_D^2}} &= \frac{0.8\times 3}{\beta_D - 0.8} \\ \Rightarrow 1.64\beta_D^2 - 1.6\beta_D &= 0 \\ \Rightarrow \beta_D &= 0.976 \end{aligned}$$

Hence, Dave must accelerate to 0.976c.

(b) What will be the elapsed time by Bob's clocks when they meet?

Now $\gamma_D = 4.56$ and $\gamma_B = 5/3$. Hence the total time is

$$t_{tot} = 3 + \gamma_D \tau = 3 + 13.67 = 16.67.$$

Thus the proper time elapsed for Bob is

$$\tau_{Bob} = \frac{t}{\gamma_B} = 10yrs$$

Hence, when they meet, Bob will be four years older than Dave.

(c) How far will they each have travelled when they meet, measured in Dave's original inerital reference frame?

When they meet, they will both have tavelled a distance of $t_{tot} \times v_B = 13.34$ lt yrs.

(d) Draw the world lines for Bob and Dave on a spacetime diagram and compare it with Figure B-2. Notice that the present scenario is the mirror image of the one discussed in connection with that figure: there Dave turned out to be four years older than Bob when they reconvened; here Bob will be four years older than Dave.

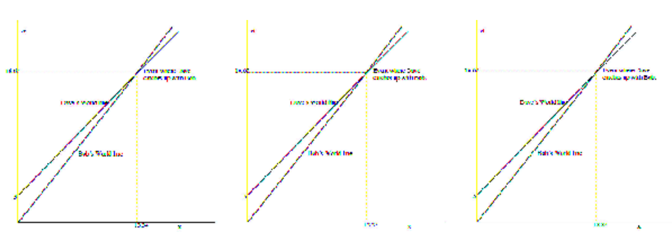

Figure 7: Problem 6 (d)

15. Rapidity (5 points)

The "rapidity" is a useful substitute for speed when a particle is moving very close to the speed of light. The rapidity, η, is defined by

$$\beta = \frac{v}{c} = \tanh \eta.$$

Remember, $\tanh x$ is the "hyperbolic tangent",

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

[Hint: in this problem you will have to know how to find the *inverse hyperbolic tangent*. You can use a calculator, a program like Maple or Mathematica, or you can solve for β in the equation above and express it in terms of logarithms involving η.]

(a) What is the speed of a particle (in units of c) that has $\eta = 1, 2, 3$, and 10?

- $\eta = 1$, $v = 0.7616c$,
- $\eta = 2$, $v = 0.9640c$ and
- $\eta = 3$, $v = 0.9951c$.
- For $\eta = 10$, $e^{-20} \approx 2.06 \times 10^{-9}$. Using the binomial expansion:

$$\beta = \tanh \eta = \frac{1 - e^{-2\eta}}{1 + e^{-2\eta}} \approx 1 - 2e^{-2\eta}.$$

This gives $v = (1 - 4.12 \times 10^{-9})c$.

(b) A proton has rest energy $m_p c^2 = 938$ MeV. What is the total energy of a proton with $\eta = 1, 2, 3,\ 10$?

The total energy is $E = \gamma m_p c^2$. In terms of the rapidity η, $\gamma = \cosh \eta$. Thus,

- $\eta = 1$, $\gamma = 1.54308$ and $E = 1.45$ GeV,
- $\eta = 2$, $\gamma = 3.7622$ and $E = 3.53$ GeV,
- $\eta = 3$, $\gamma = 10.0677$ and $E = 9.44$ GeV and
- $\eta = 10$, $\gamma = 11013.2$ and $E = 1.03 \times 10^4$ GeV.

(c) A proton has *kinetic* energy (K.E.$= E - mc^2$) of 1 GeV. What is its rapidity?

$E = 1938$ MeV. Thus $\gamma = \frac{E}{mc^2} = \frac{1938}{938} \approx 2.06$. This gives the rapidity $\eta = \cosh^{-1} \gamma \approx 1.35$.

16. Rapidities Add (5 points)

All the motion in this problem is collinear — say along the x axis. A particle moves with velocity u' along the x' axis in the frame Σ'. This frame, in turn, moves with velocity v along the x axis in the frame Σ. The particle's velocity observed in Σ is u.

This is the standard configuration for velocity addition. In lecture (and in the texts) we derived:

$$u = \frac{u' + v}{1 + u'v/c^2}.$$

Let η, ξ, and ξ' be the rapidities corresponding to v, u, and u', respectively. (So $v = c\tanh\eta$, $u = c\tanh\xi$, and $u' = c\tanh\xi'$).

Show, as stated in lecture, that $\xi = \eta + \xi'$. So in special relativity (parallel) rapidities add.

In this solution, the velocity in the Σ' frame is chosen to be u'. Using the given relations between the rapidities and velocities, the velocity addition law can be re-written as

$$\tanh\xi = \frac{\tanh\xi' + \tanh\beta}{1 + \tanh\xi'\tanh\beta}.$$

>From the definition of $\cosh x$ and $\sinh x$

$$\begin{aligned}
\cosh(a+b) &= \frac{1}{2}(e^{a+b} + e^{-(a+b)}) \\
&= \frac{1}{4}(e^{a+b} + e^{-(a+b)} + e^{a+b} + e^{-(a+b)}) \\
&= \frac{1}{4}(e^{a+b} + e^{a-b} + e^{-(a+b)} - e^{-a+b} + e^{a+b} - e^{-a+b} + e^{-(a+b)} + e^{-a+b}) \\
&= \frac{1}{4}((e^a + e^{-a})(e^b + e^{-b}) + (e^a - e^{-a})(e^b - e^{-b})) \\
&= \cosh a\cosh b + \sinh a\sinh b.
\end{aligned}$$

Similarly

$$\sinh(a+b) = \cosh a\sinh b + \sinh a\cosh b.$$

These give

$$\tanh(a+b) = \frac{\sinh a + b}{\cosh a + b} = \frac{\tanh a + \tanh b}{1 + \tanh a\tanh b}.$$

Thus

$$\begin{aligned}
\tanh\xi &= \tanh(\xi' + \beta) \\
\Rightarrow \xi &= \xi' + \beta
\end{aligned}$$