Physics 8.20 Special Relativity IAP 2008

Problem Set # 4 Solutions

1. **Proper acceleration** (4 points)

Note: this problem and the next two ask you to work through the details of results derived in lecture. Let a particle be moving along the x-axis when viewed in the frame Σ. The particle's *proper acceleration*, α, is defined as its acceleration measured in its instantaneous rest frame. Specifically, suppose at time t the particle has velocity v. Then it is instantaneously at rest in the frame Σ' moving with velocity v relative to Σ. Then $\alpha = dv'/dt'$.

Show that the acceleration observed in Σ is related to α by

$$\frac{dv}{dt} = \gamma^{-3}(v)\alpha$$

where $\gamma(v) = 1/\sqrt{1 - v^2/c^2}$ as usual, and v is the instantaneous velocity.

`If the frame` Σ' `is moving with velocity` $v\hat{x}$ `relative to` Σ`, then`

$$t = \gamma(t' + \beta x'/c)$$

$$\Rightarrow dt = \gamma dt'(1 + u'_x v/c^2)$$

`and,`

$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2}$$

$$\Rightarrow du_x = \frac{du'_x}{\gamma^2(1 + u'_x v/c^2)^2}\,.$$

`So the acceleration of a particle in the two frames is related by`

$$a_x = \frac{du_x}{dt} = \frac{a'_x}{\gamma^3(1 + u'_x v/c^2)^3}\,.$$

`Now if` Σ' `is the instantaneous rest frame of the particle,` $u'_x = 0$, $v = u_x$ `and` $a'_x \equiv \alpha$. `So,`

$$a_x = \frac{\alpha}{\gamma^3}\,.$$

2. Constant proper acceleration (5 points)

Suppose a particle experiences constant proper acceleration, $\alpha = \alpha_0$. Suppose it starts out at rest in Σ at $t = 0$.

(a) Use the result of the previous problem to show that its velocity (as measured in Σ) is given by

$$v(t) = \frac{\alpha_0 t}{\sqrt{1 + (\frac{\alpha_0 t}{c})^2}}$$

`>From problem 1,`

$$\frac{dv}{dt} = \alpha_0 (1 - v^2/c^2)^{3/2}$$

$$\Rightarrow \frac{d\beta}{(1-\beta^2)^{3/2}} = \frac{\alpha_0}{c} dt$$

$$\Rightarrow \frac{\beta}{\sqrt{1-\beta^2}} = \frac{\alpha_0 t}{c}$$

$$\Rightarrow v(t) = \frac{\alpha_0 t}{\sqrt{1 + (\alpha_0 t/c)^2}} \, .$$

(b) Let $\alpha_0 = g = 9.8\text{m/sec}^2$. How long would it take the particle to reach $v = 0.99c$?, $v = 0.999c$? according to an observer in Σ?

`>From part (a),`

$$t = \frac{c}{\alpha_0} \frac{\beta}{\sqrt{1-\beta^2}} \, .$$

`So, time taken to reach` $\beta = 0.99$ `is about 6.8 years.`
`And time taken to reach` $\beta = 0.999$ `is about 21.7 years.`

(c) How long would it take to reach these speeds according to an observer on the particle?

`The observer on the particle measures proper time.`

$$\begin{aligned} dt &= \gamma d\tau \\ \Rightarrow \tau &= \int_0^t d\tilde{t} \sqrt{1-\beta^2} \\ &= \int_0^t d\tilde{t} \frac{1}{\sqrt{1 + \alpha_0^2 \tilde{t}^2/c^2}} \\ &= \frac{c}{\alpha_0} \sinh^{-1} \frac{\alpha_0 t}{c} \end{aligned}$$

So, the proper time taken to reach $\beta = 0.99$ is about 2.6 years and the proper time taken to reach $\beta = 0.999$ is about 3.7 years (using the results of part (b)).

3. Hyperbolic space travel I (7 points)

(a) Take the result of the previous problem,

$$v(t) = \frac{\alpha_0 t}{\sqrt{1 + (\frac{\alpha_0 t}{c})^2}}$$

and integrate $v(t) = dx/dt$ to obtain an expression for $x(t)$. Take $\alpha = g =$10m/sec^2 and show that you reproduce the result obtained in class:

$$X(T) = \sqrt{T^2 + 1} - 1$$

where T is in years and X in light-years.

Using $v(t) = dx/dt$,

$$dx = \frac{\alpha_0 t}{\sqrt{1 + (\frac{\alpha_0 t}{c})^2}} dt.$$

Substituting $\alpha_0 t/c = \sinh\chi$:

$$dx = \frac{c^2}{\alpha_0} \frac{\sinh\chi \cosh\chi d\chi}{\sqrt{1 + \sinh^2\chi}} = \frac{c^2}{\alpha_0} \sinh\chi d\chi.$$

Integrating:

$$\begin{aligned} x(\chi_2) - x(\chi_1) &= \frac{c^2}{\alpha_0}(\cosh\chi_2 - \cosh\chi_1) \\ &= \frac{c^2}{\alpha_0}(\sqrt{1 + \sinh^2\chi_2} - \sqrt{1 + \sinh^2\chi_1}) \\ \Rightarrow x(t_2) - x(t_2) &= \frac{c^2}{\alpha_0}(\sqrt{1 + (\frac{\alpha_0 t_2}{c})^2} - \sqrt{1 + (\frac{\alpha_0 t_1}{c})^2}). \end{aligned}$$

As $\alpha_0 = g = 10m/s$, $c/\alpha_0 = 3 \times 10^7 s = 1$ yr. Then $T = \alpha_0 t/c$ is in year units and $X = x\alpha_0/c^2$ is light year units. This gives the result obtained in class:

$$X(T) = \sqrt{T^2 + 1} - 1.$$

(b) Likewise confirm the result from lecture that relates the passage of proper time (experienced by the astronauts), τ, and the passage of time in the rest frame from which the rocket originated:

$$T = \sinh\tau$$

— again both times in years.

`>From the value of` v `in (a),`

$$1 - v(t)^2 = \frac{1}{1 + (\frac{\alpha_0 t}{c})^2} = \frac{1}{\gamma(v)}.$$

`But` $d\tau = dt/\gamma$. `Again substituting` $\alpha_0 t/c = \sinh\chi$,

$$d\tau = \frac{c}{\alpha_0} d\chi.$$

`This gives` $T = \sinh\tau$, `where` τ `is now measured in yrs.`

(c) Now suppose that the space travellers take a more "realistic" journey where they accelerate at g for the first half of the trip and decelerate at g for the second half. Find expressions for

i. How far can they travel in a time T (measured in the originating frame)?

`Here, for the first half, the accelaration is positve, and then negative. Thus the velocity is`

$$\begin{aligned} v(t) &= \frac{\alpha_0 t}{\sqrt{1 + (\frac{\alpha_0 t}{c})^2}}, 0 \le t \le T/2 \\ &= \frac{\alpha_0 (T - t)}{\sqrt{1 + (\frac{\alpha_0 (T-t)}{c})^2}}, T/2 \le t \le T. \end{aligned}$$

`Using the integrated equation for` x `in part (a) with` $t_2 = T/2$, $t_1 = 0$ `and then integrating the same way for` $t_2 = T$, $t_1 = T/2$. `By a change of variable` $x = T - t$, `for the second integration you can easily see that the itegral will be equal to the integration of` $v(t)dt$ `for the part of the trip we get:`

$$x(T) - x(0) = x(T) - x(T/2) + x(T/2) - x(0) = 2 \times (x(T/2) - x(0)) = \sqrt{T^2 + 4} - 2;$$

ii. How much proper time, τ, passes on a journey that takes T years (as observed in the originating frame).

$$
\begin{aligned}
\gamma &= \sqrt{1+(\frac{\alpha_0(t)}{c})^2}, 0 \le t \le T/2 \\
&= \sqrt{1+(\frac{\alpha_0(T-t)}{c})^2}, T/2 \le t \le T. \\
\Rightarrow \tau &= \int_0^{T/2} \gamma^{-1} dt + \int_{T/2}^{T} \gamma^{-1} dt \\
&= 2\int_0^{T/2} \frac{1}{\sqrt{1+T^2}} dT \\
&= 2\sinh^{-1}\frac{T}{2} \\
\Rightarrow T = 2\sinh\frac{\tau}{2}.
\end{aligned}
$$

4. Hyperbolic space travel II (7 points)

Assume that a rocket can produce an acceleration $g_0 = 9.8\text{m/sec}^2$. [As described in lecture, this acceleration is approximately the same as that due to gravity at the surface of the earth. Astronauts will be able to live comfortably in this spaceship, as if in earth's gravity.) Assume, in addition, that in travelling to any destination the rocket *will accelerate half the way and decelerate during the second half of the journey.*

(a) Calculate the travel time *as measured by the space traveller* to the moon. (Assume the moon is at a distance of 382,000 km.) Compare with the Galilean answer.

For a rocket with constant proper acceleration α_0, we derived the following result in Problem 2 :

$$v(t) = \frac{\alpha_0 t}{\sqrt{1+(\alpha_0 t/c)^2}},$$

where v, t are measured in the earth frame.
We integrate this to obtain a hyperbolic equation for the earth-measured distance of travel, d, as a function of time:

$$d(t) = \frac{c^2}{\alpha_0}\sqrt{1+(\alpha_0 t/c)^2} - \frac{c^2}{\alpha_0}.$$

We can solve for t to get

$$t = \sqrt{d\left(\frac{d}{c^2} + \frac{2}{\alpha_0}\right)}.$$

Convert the earth time to the astronaut's time using

$$\tau = \frac{c}{\alpha_0}\sinh^{-1}\frac{\alpha_0 t}{c}$$

derived in Problem 2.
So, for travel halfway to the moon with $\alpha_0 = 9.8m/s^2$,

$$d = \frac{1}{2} \times 3.82 \times 10^8 m$$

$$\Rightarrow t = 1.734 \text{ hrs}$$

$$\Rightarrow \tau = 1.734 \text{ hrs}$$

Twice the halfway travel time gives the total travel time of

$$\boxed{\text{3.47 hrs}}.$$

A Galilean calculation would give a halfway travel time (in all frames) of

$$\sqrt{\frac{2d}{\alpha_0}} = 1.734\, hrs$$

which gives a total time of

$$\boxed{\text{3.47 hrs}}.$$

For such a short journey, there is no difference between Galilean, earth and astronaut travel times.

(b) Answer the same questions for travel to Neptune, assumed to be at a distance of 4.5×10^9 km.

For travel halfway to Neptune,

$$d = \frac{1}{2} \times 4.5 \times 10^{12} m$$

$$\Rightarrow t = 7.843 \text{ days}$$

$$\Rightarrow \tau = 7.842 \text{ days}$$

So the total travel time is

$$\boxed{\text{15.7 days}}.$$

A Galilean calculation would give a total travel time (in all frames) of

$$\boxed{\text{15.7 days}}.$$

So once again, the journey is short enough to not produce any significant difference between the Galilean, earth and astronaut travel times.

(c) Answer the same questions for travel to Alpha Centauri, assumed to be at a distance of 4.3 light years away. What is the velocity of the rocket, in the earth's reference frame, at the half way point of the journey?

For travel halfway to Alpha Centauri,

$$d = \frac{1}{2} \times 4.3 \text{ lyrs}$$

$$\Rightarrow t = 2.96 \text{ yrs}$$

$$\Rightarrow \tau = 1.78 \text{ yrs}$$

So the total travel time is

$$\boxed{3.56 \text{ yrs}}.$$

At the halfway point,

$$\beta = \frac{\alpha_0 t/c}{\sqrt{1 + (\alpha_0 t/c)^2}} = \boxed{0.95}$$

A Galilean calculation would give a total travel time (in all frames) of

$$\boxed{4.08 \text{ yrs}}.$$

Now the earth frame travel time is larger than the Galilean time which in turn is larger than the astronaut's time.

(d) What is the value of β that would enable a second astronaut, travelling at constant speed, to travel from the earth to Alpha Centauri in the same travel time as that taken by the rocket described above?

The rocket described above takes 5.90 yrs (as measured on earth) to travel to Alpha Centauri 4.3 lyrs away. This corresponds to an average speed of

$$\frac{4.3c}{5.92} = \boxed{0.726 \text{ c}}$$

and a rocket traveling at that constant speed will complete the journey in the same amount of time.

5. Hyperbolic space travel III (8 points)

Aliens arrive on Earth and tell us that they have come from very far away. They had been watching light emitted from the primitive Earth and saw that life was evolving. They took a risk and headed off for Earth. Their journey took them a distance of 100,000,000 light-years. After getting our units right, we figured out that only 5 years of proper time elapsed for them during their great journey. Being smart aliens, they travelled hyperbolically — they accelerated at the natural acceleration of gravity on their planet, G, for the first half, and then decelerated for the second half of the journey.

How strong is gravity on the surface of their planet? (Hint: To solve the transcendental equation for G, you could graph both sides as a function of G; or start with a particular value for G and iterate; or use a symbolic manipulation package like Matlab

or Mathematica, both available on Athena.)

In problem 3 we solved for $x(t)$, with acceleration α_0. In this problem the acceleration is G. Suppose the whole trip takes time T, then the midpoint is reached at $T/2$. Thus

$$\begin{aligned}
x(\frac{1}{2}T) &= \frac{c^2}{G}(\sqrt{1+(\frac{GT}{2c})^2}-1) \\
\Rightarrow \frac{G}{c^2}x(\frac{1}{2}T) &= \sqrt{1+\sinh^2\frac{G\tau}{2c}}-1 \\
&= \cosh\frac{G\tau}{2c}-1 \\
&= 2\sinh^2\frac{G\tau}{4c} \\
\Rightarrow \frac{G}{g}\times\frac{gx(T/2)}{c^2} &= 2\sinh^2\frac{G}{g}\times\frac{g\tau}{4c}.
\end{aligned}$$

As $c/g \approx 1$ yr, we can now insert the values given in lt years and years. Call $2.5G/2g = G'$. Then, inserting the values

$$\begin{aligned}
2G'\times 10^7 &= \sinh^2 G' \\
\Rightarrow \sqrt{2G'}\times 10^{3.5} &= \sinh G' \\
&\approx \frac{e^{G'}}{2} \\
\Rightarrow G' &= \frac{3}{2}\ln 2 + 3.5\ln 10 + \frac{1}{2}\ln G'.
\end{aligned}$$

Solving by iteration: Choose on right hand side

$$\begin{aligned}
G' = 1 &\Rightarrow G' = 9.098; \\
G' = 9.098 &\Rightarrow G' = 10.202; \\
G' = 10.202 &\Rightarrow G' = 10.261; \\
G' = 10.261 &\Rightarrow G' = 10.262; \\
G' = 10.262 &\Rightarrow G' = 10.263.
\end{aligned}$$

Thus $G' = 10.26$, which gives $G = 80.5\,m/s^2$.

We could also plot the left and right hand sides of the equations as functions of G' and look at their point of intersection.

6. Non-relativistic limit (3 points)

The correct relativistic formula for the energy of a particle of rest mass m and speed v is $E(v) = m\gamma(v)c^2$.

(a) Expand $E(v)$ to order v^4. Identify the first correction to the usual non-relativistic formula for the kinetic energy, $T = \frac{1}{2}mv^2$.

`The relativistic expression for the energy of a particle of rest mass` m `and speed` v `is`

$$\begin{aligned} E(v) &= \gamma(v)mc^2 \\ &= mc^2\left(1 + \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4 + \ldots\right) \\ &= mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2} + \ldots \end{aligned}$$

`So the first relativistic correction to the kinetic energy is`

$$\Delta T = \frac{3}{8}m\frac{v^4}{c^2}\,.$$

(b) How large is the correction of part a) for the Earth relative to the Sun? In other words, approximately how much heavier is the Earth on account of its motion around the Sun? You will need the Earth's mass and its orbital speed to complete this estimate.

`The earth's mass is` $m = 6\times10^{24}\,kg$ `and its orbital speed is` $v = 30\,km/s$`.`
`The leading order correction to mass is given by`

$$\begin{aligned} \Delta m^{(0)} &= \frac{mv^2}{2c^2} \\ &= \frac{1}{2}m\beta^2 \\ &= 3\times10^{16}kg \end{aligned}$$

`This is correction by`

$$\left(\frac{\Delta m}{m}\times 100\right) \quad \texttt{or} \quad 5\times10^{-7}\%\,.$$

`The correction to galilean kinetic energy calculated in part (a) contributes a mass of (next to leading order correction)`

$$\begin{aligned} \Delta m^{(1)} &= \frac{\Delta T}{c^2} \\ &= \frac{3}{8}m\beta^4 \\ &= 2.25\times10^{8}kg \end{aligned}$$

`This is a correction by`

$$\left(\frac{\Delta m}{m}\times 100\right) \quad \texttt{or} \quad 4\times10^{-15}\%\,.$$

(c) How large a percentage error did you make by using the approximation of part a) instead of the exact result in the case of the Earth's motion around the Sun?

`The exact relativistic correction to the earth's mass is given by`

$$\begin{aligned}\Delta m_{ex} &= \gamma m - m \\ &= m\left(\frac{1}{2}\beta^2 + \frac{3}{8}\beta^4 + \frac{5}{16}\beta^6 + \frac{35}{128}\beta^8 + \ldots\right).\end{aligned}$$

`In approximation used in part (b) we just took first two terms into account i.e.` $\Delta m = \Delta m^{(0)} + \Delta m^{(1)}$. `Percentage error due to this is given by`

$$\frac{\Delta m_{ex} - \Delta m}{m} \times 100\% = \left(\frac{5}{16}\beta^6 + \frac{35}{128}\beta^8 + \ldots\right) \approx 3.1 \times 10^{-23}$$

7. Hard work (2 points)

A proton initially moves at $v = 0.99c$ in some reference frame.

(a) How much work must be done on the proton to increase its speed from $0.99c$ to $0.999c$?

`The proton mass is` $m_p = 939\ MeV/c^2$. `The work done in increasing its speed from` v_1 `to` v_2 `is`

$$W = \left(\frac{1}{\sqrt{1-\beta_1^2}} - \frac{1}{\sqrt{1-\beta_2^2}}\right) \times 939\ MeV\,.$$

`(a) If` $v_1 = 0.999c$ `and` $v_2 = 0.99c$, `then`

$$W \approx 14.3\ \texttt{GeV}\,.$$

(b) How much work must be done to increase its speed from $0.999c$ to $0.9999c$?

`If` $v_1 = 0.9999c$ `and` $v_2 = 0.999c$, `then`

$$W \approx 45.4\ \texttt{GeV}\,.$$

8. Solar Power (5 points)

The net result of the fusion reaction that fuels the sun is to turn four protons and two electrons into one helium nucleus,

$$4p^+ + 2e^- \rightarrow {}^4He^{++}.$$

Other particles are given off (neutrinos and photons), but you can assume they eventually show up as energy. The masses of the relevant nuclei are as follows:

$m_p = 1.6726 \times 10^{-27}$ kg.
$m_e = 9.1094 \times 10^{-31}$ kg.
$m_{\,{}^4He} = 6.6419 \times 10^{-27}$ kg.

(a) How much energy is released when a kilogram of protons combines with just enough electrons to fuse completely to form helium?

`4 protons fuse with 2 electrons to release`

$$(4m_p + 2m_e - m_{He})c^2 = 4.5290 \times 10^{-12} J\,.$$

`A kilogram of protons contains`

$$4 \times 1.4947 \times 10^{26} \text{ protons}$$

`which release`

$$1.4947 \times 10^{26} \times 4.5290 \times 10^{-12} J = \boxed{6.7694 \times 10^{14} \text{ J}}.$$

(b) How many kilograms of methane would you have to burn to produce the same amount of energy? (Go to your favorite source of chemistry information (the Web, perhaps, or the CRC Handbook) and determine the reaction by which methane burns and how much energy is released per gram molecular weight of methane. The usual units are KCal/mole).

`We can calculate that 1000 kg of methane burns to produces how much energy by looking up the values in CRC. The reaction is`

$$CH_4 + 2O_2 \rightarrow CO_2 + H_2O$$

`which releases an energy of 212.8 Kcal/mole. So for 1000 kg, this gives:`

$$212.8 \times 10^3 \text{cal/mole} \times \frac{1000}{0.016} \text{ moles} \times 4.2 \text{ J/cal} = 5.586 \times 10^{10} J$$

`Dividing by` c^2`, the mass equivalent of the above energy is`

$$\boxed{\text{0.6207 micro-grams}}.$$

`So the amount of methane that will produce the same amount of energy as that produced by fusing 1 kg of protons:`

$$1000 \times \frac{6.7694 \times 10^{14}}{5.586 \times 10^{10}} \text{ kg} = \boxed{1.2119 \times 10^7 \text{ kg}}.$$

9. $\mathbf{E = mc^2}$ (2 points)

Assume the heat capacity of water is equal to 4.2 joules/g-K, and constant as a function of temperature. How much does the mass of a ton of water increase when it is heated from freezing (0°C) to boiling (100°C)?

The amount of heat energy transfered to 1000 kg of water to raise its temperature from 0° C to 100° C is

$$10^6 \text{ g} \times 4.2 \text{ J/(° g)} \times 100° = 4.2 \times 10^8 J\,.$$

This is equivalent to a mass of

$$\boxed{4.67 \times 10^{-9} \text{ kg}}\,.$$

10. Enormous energies (2 points)

Quasars are the nuclei of active galaxies in the early stages of their formation. A typical quasar radiates enery at the rate of 10^{41} watts. At what rate is the mass of this quasar being reduced to supply this energy? Express your answer in solar mass units per year, where one solar mass unit, 1 smu = 2×10^{30} kg, is the mass of our sun.

The mass of the quasar is reduced at a rate of

$$\frac{10^{41}}{9 \times 10^{16}} \text{ kg/s} \times \frac{1}{2 \times 10^{30}} \text{ smu / kg} \times (3600 \times 24 \times 365) \text{ s/yr} = \boxed{17.52 \text{ smu/yr}}\,.$$

11. Classical physics and the speed of light (2 points)

(a) How much energy would it take to accelerate an electron to the speed of light according to "classical" (before special relativity) physics?

If an electron is accelerated from rest to the speed of light, then classically it would require

$$\frac{1}{2} m_e c^2 = \boxed{4.1 \times 10^{-14} \text{ J}}\,.$$

(b) With this energy what would its actual velocity be?

`If the above amount of energy is given to an electron then`

$$T = (\gamma - 1)m_e c^2 = \frac{1}{2} m_e c^2$$

$$\Rightarrow \gamma = \frac{3}{2}$$

$$\Rightarrow \boxed{\beta = 0.745}.$$

12. A useful approximation (4 points)

(a) Show that for an extremely relativistic particle, the particle speed u differs from the speed of light c by

$$\Delta u = c - u = \frac{c}{2}\left(\frac{m_0 c^2}{E}\right)^2$$

where m_0 is the rest mass and E is the energy.

$$E = \gamma m_0 c^2$$

$$\Rightarrow \beta = \sqrt{1 - \left(\frac{m_0 c^2}{E}\right)^2}$$

$$\Rightarrow \beta \approx 1 - \frac{1}{2}\left(\frac{m_0 c^2}{E}\right)^2 \quad \texttt{for a particle with } E >> m_0 c^2$$

$$\Rightarrow \Delta u = c - u \approx \frac{c}{2}\left(\frac{m_0 c^2}{E}\right)^2$$

(b) Find Δu for electrons produced by

i. MIT's Bates Accelerator Center, where $E = 900$ MeV.

`E = 900 MeV at Bates Accelerator Center gives` Δu = $\boxed{\texttt{48.4 m/s}}$.

ii. The Jeffreson Lab (in Newport News, Virginia), where $E = 12$ GeV.

`E = 12 GeV at Jefferson Lab gives` Δu = $\boxed{\texttt{0.272 m/s}}$.

iii. The Stanford Linear Accelerator Center (in Palo Alto, California), where $E = 50$ Gev.

`E = 50 GeV at SLAC gives` Δu `=` $\boxed{\texttt{0.0157 m/s}}$.

13. Pressure of light (5 points)

French §6, Problem 6-7, page 201: A photon rocket

`Four momentum of the system BEFORE collision:`

$$P = M_i(c, 0)$$

`Four momentum of the system AFTER collision:`

$$P' = \frac{E}{c}(1, -1) + M_f\gamma(c, v)$$

`Where E is the energy of photons.`

`>From conservation of four-momentum:`

$$P = P'$$

`we get two equations:`

$$M_i c = E/c + M_f\gamma c$$
$$0 = -E/c + M_f\gamma v$$

`Eliminate` E `from above:`

$$M_i c = M_f\gamma v + M_f\gamma c$$
$$\Rightarrow M_i/M_f = \gamma(1+\beta) = \sqrt{\frac{1+\beta}{1-\beta}}$$

French §6, Problem 6-9, page 201: A laser beam

(a) `Energy radiated by laser per second`

$$P_0 = 10^{20}\, s^{-1}\frac{hc}{\lambda} = 33.1\, W$$

`Since laser is at rest w.r.t earth therefore this is also the rate at which observer on earth observes the power radiated.`

(b) Let E_γ be the total energy radiated by laser in form of photons. By energy conservation in earth's frame

$$Mc^2 = \gamma(M - \Delta M)c^2 + E_\gamma\,,$$

and by momentum conservation

$$0 = -\frac{E_\gamma}{c} + (M - \Delta M)\gamma v.$$

Eliminating E_γ from above two equation we get

$$Mc^2 = \gamma(M - \Delta M)c(c + v)$$

which simplifies to

$$\frac{M - \Delta M}{M} = \sqrt{\frac{1-\beta}{1+\beta}}.$$

Expanding in powers of beta upto leading term we get

$$\beta = \frac{\Delta M}{M} \Rightarrow v = \frac{\Delta M}{M}c.$$

Now

$$\Delta M = 10^{20} \times \frac{hc}{\lambda} \times 10\,yrs/c^2 = 1.16 \times 10^{-7}\,kg.$$

This gives

$$v = 3.48\,m/s.$$

(c) Incident energy of each photon when laser is moving with velocity βc

$$E_\gamma = \sqrt{\frac{1-\beta}{1+\beta}}\,E'_\gamma$$

where E'_γ is energy of photon in laser frame. Similarly rate at which photon are observed on earth's frame is given by

$$r = \sqrt{\frac{1-\beta}{1+\beta}}\,r'$$

where $r' = 10^{20}\,s^{-1}$. Therefore observed power on earth as a function of beta is given by

$$P = E_\gamma r = \frac{1-\beta}{1+\beta}E'_\gamma r' = \frac{1-\beta}{1+\beta}P_0.$$

Taking $P = P_0 - \Delta P$ and expanding in power of β we get

$$\Delta P = 2P_0\beta = 2 \times 33.1 \times 1.16 \times 10^{-8}\,W = 7.7 \times 10^{-7}\,W\,.$$

So the rate seen on earth after 10 years has decreased by $7.7 \times 10^{-7}\,W$.

(d) Loss in incident power is accounted for by energy used in accelerating the laser.

14. Relativistic collisions and decays (17 points)

Note that many of these problems are easiest if you use invariants.

French §6, Problem 6-5, page 200: Three particle decay

(a) `Four momentum Before the decay:`

$$P = M_0 c(1, 0, 0)$$

`Four momentum after the decay:`

$$P' = m_0\gamma_1 c(1, -\beta_1, 0) + m_0\gamma_2 c(1, 0, -\beta_2) + m_0\gamma_3 c(1, \beta_3\cos\theta, \beta_3\sin\theta)$$

`Where we know:`

$$\beta_1 = 4/5$$
$$\gamma_1 = 5/3$$
$$\beta_2 = 3/5$$
$$\gamma_2 = 5/4$$

`>From the conservation of four-momentum` $P = P'$ `we'll get three equations:`

$$M_0 = m_0(\gamma_1 + \gamma_2 + \gamma_3) \tag{1}$$
$$\gamma_1\beta_1 = \gamma_3\beta_3\cos\theta \tag{2}$$
$$\gamma_2\beta_2 = \gamma_3\beta_3\sin\theta \tag{3}$$

`Devide (2) and (3):`

$$\tan\theta = \frac{\gamma_2\beta_2}{\gamma_1\beta_1} = 9/16$$

$$(2) \Rightarrow \gamma_3\beta_3 = \frac{\gamma_1\beta_1}{\cos\theta} = \gamma_1\beta_1\sqrt{1+\tan^2\theta} = \frac{\sqrt{337}}{12} \Rightarrow \beta_3^2 = \frac{337}{337+144}$$

$$\beta_3 = 0.84$$

(b) `>From (1) we have:`

$$M_0/m_0 = \gamma_1 + \gamma_2 + \gamma_3 = 1.67 + 1.25 + 1.84 = 4.76$$

French §6, Problem 6-11, page 201: Pion decay

(a) γ of the π^0 is:

$$\gamma = \frac{1\ \ GeV + 135\ \ MeV}{135\ \ MeV} = 1135/135 = 8.4 \Rightarrow \beta = 0.9929$$

Before the decay we have the four-momentum:

$$m\gamma c(1, \beta)$$

After the decay we have the two photons:

$$(E/c, E/c) + (E'/c, -E'/c)$$

>From the law of conservation of four-momentum we get:

$$E + E' = m\gamma c^2$$

$$E - E' = m\gamma\beta c^2$$

$$E = \frac{1}{2}mc^2\gamma(1+\beta) = 1131\ \text{MeV}$$

$$E' = \frac{1}{2}mc^2\gamma(1-\beta) = 4\ \text{MeV}$$

(b) This time we'll have:

$$E/c(1, \cos\theta, \sin\theta) + E'/c(1, +\cos\theta, -\sin\theta) = m\gamma c(1, \beta, 0)$$

The last component of the above equation will give:

$$E = E'$$

>From the zeroth component (energy conservation) we'll get:

$$2E/c = m\gamma c$$

>From the 1st component we'll get:

$$2E/c\cos\theta = m\gamma c\beta$$

After deviding the above two equations we get:

$$\cos\theta = \beta \Rightarrow \theta = 0.1192\ \text{Radians} = 6.8^\circ$$

The angle between the two gamma rays is twice the above angle , i.e. 13.6°.

French §6, Problem 6-14, page 202: Impossible processes

Factors of c are suppressed in this calculation.

(a) A single photon (with four-momentum $p_\gamma = (E_\gamma, E_\gamma, 0, 0)$) strikes a stationary electron ($p_i = (m_e, 0, 0, 0)$) and gives up all its energy to the electron (which has four-momentum p_f). From conservation of four-momentum,

$$p_\gamma + p_i = p_f \,.$$

Squaring both sides,

$$m_e^2 = m_e^2 + 2p_\gamma \cdot p_i$$
$$\Rightarrow m_e E_\gamma = 0$$

which is impossible unless $E_\gamma = 0$ (i.e. there is no photon).

(b) A single photon (with four-momentum $p_\gamma = (E_\gamma, E_\gamma, 0, 0)$) in empty space is transformed into an electron and a positron with four-momenta p_1 and p_2 respectively. So,

$$p_\gamma = p_1 + p_2$$
$$\Rightarrow 0 = 2m_e^2 + 2p_1 \cdot p_2$$
$$\Rightarrow p_1 \cdot p_2 = -m_e^2$$

which is impossible because the scalar product of four-momenta corresponding to massive particles has to be greater than the product of the masses (work it out in the center of mass frame to convince yourself of this). Another way to deduce that this process is impossible is to work in the center of mass frame of the electron-positron pair. Then the photon must have zero momentum to conserve momentum but that would mean that energy cannot be conserved.

(c) A fast positron and a stationary electron annihilate, producing only one photon. This is the reverse process of (b) (Lorentz transformed to the rest frame of the electron) and so is impossible because (b) is impossible.

French §6, Problem 6-15 (a), page 202: Proton Collision

Lorentz factor for incident proton is $\gamma = (m_p + K)/m_p = 1.46$ which gives $v = 0.731\,c$. From energy conservation we get

$$2m_p + K = 2E'$$

where E' is energy of each final proton. This gives $E' = 1158.5\,MeV$. Therefore $\gamma' = 1158.5/940 = 1.23$ which gives $v' = 0.585\,c$. From momentum conservation

$$\gamma m_p v = 2E'v'\cos\theta$$

$$\Rightarrow \cos\theta = \frac{(m_p + K)v}{2E'v'} = 0.7426$$

which gives $\theta = 42.045°$. Therefore inclusive angle is $2\theta = 84.09°$.

French §6, Problem 7-1, page 225: Kaon decay

Let's denote P_1 the four-momentum of the K meson, P_2 the moving π and P_3 the π at rest.

$$P_1 = m_K\gamma_K c(1, \beta_K)$$
$$P_2 = m_\pi\gamma_\pi c(1, \beta_\pi)$$
$$P_3 = m_\pi c(1, 0)$$

We have:

$$P_1 = P_2 + P_3 \Rightarrow (P_1 - P_3)^2 = P_2^2$$
$$m_K^2c^2 + m_\pi^2c^2 - 2m_\pi c^2 m_K\gamma_K = m_\pi^2c^2$$
$$\Rightarrow \gamma_K = \frac{m_K}{2m_\pi} = 1.8$$
$$E_K = m_K\gamma_K c^2 = 889 \text{ MeV}$$
$$K_K = 889 - 494 = 395 \text{ MeV}$$

from Energy conservation:

$$E_\pi = 889 - 137 = 752 \text{ MeV}$$

$$K_\pi = 752 - 137 = 615 \text{ MeV}$$

French §6, Problem 7-2, page 225: Pair production

For the minimum energy of the gamma ray, the two e^- and the e^+ will be at rest in their CENTER of MASS frame so their total four-momentum will look like $(3m_ec, 0)$ since P^2 is invariant under lorentz transformation we'll have:

$$[E/c(1,1) + m_ec(1,0)]^2 = 9m_e^2c^2$$
$$0 + m_e^2c^2 + 2m_eE = 9m_e^2c^2$$
$$E = 4m_ec^2 = 2.04 \text{ MeV}$$

French §6, Problem 7-4, page 226: elastic scattering

15. Fixed-target collisions (RH) (5 points)

(a) A proton (with rest mass m) accelerated in a proton synchrotron to a kinetic energy K strikes a second (target) proton at rest in the laboratory. The collision is entirely inelastic in that the rest energy of the two protons, plus all of the kinetic energy consistent with the law of conservation of momentum, is available to generate new particles and to endow them with kinetic energy. Show that the energy available for this purpose is given by

$$\mathcal{E} = 2mc^2\sqrt{1 + \frac{K}{2mc^2}}.$$

Before the collision, the total energy and the total momentum of the two protons is:

$$E_i = K + 2mc^2$$

$$p_i = \sqrt{K^2 + 2Kmc^2}/c$$

After the collision we can treat all the products as a single particle with mass M and velocity v. having energy and momentum:

$$E_f = \gamma Mc^2$$

$$p_f = \gamma Mv$$

By energy and momentum conservation, $E - i = E_f$ and $p_i = p_f$:

$$K + 2mc^2 = \gamma Mc^2$$

$$\sqrt{K^2 + 2Kmc^2}/c = \gamma Mv$$

DEviding the two equations gives β,

$$\beta = v/c = \frac{\sqrt{K^2 + 2Kmc^2}}{K + 2mc^2} = \sqrt{\frac{K}{K + 2mc^2}}$$

We can then find γ:

$$\gamma = \sqrt{1 + \frac{K}{2mc^2}}$$

This gives us Mc^2, the total energy available for making particles and endowing them with kinetic energy (i.e. the total energy in the frame where the momenum is zero),

$$Mc^2 = \frac{K + 2mc^2}{\gamma} = 2mc^2 = \sqrt{1 + \frac{K}{2mc^2}}$$

(b) How much energy is made available when 100-GeV protons are used in this fashion?

$$E_0 = 2(0.938\ \ GeV)\sqrt{1 + \frac{100}{2 \times 0.938}} = 13.6\ \ GeV$$

(c) What proton energy would be required to make 100 GeV available? This should be compared with the result in the next problem.

`For` $E_0 = 100\ \ GeV$`,`

$$100\ \ GeV = 2(0.938\ \ GeV)\sqrt{1 + \frac{K}{2 \times 0.938}}$$
$$\Rightarrow K = 5.33\ \ TeV$$

16. Center-of-mass collisions (RH) (5 points)

(a) In modern experimental high-energy physics, energetic particles are made to circulate in opposite directions in so-called storage rings and are permitted to collide head-on. In this arrangement, each particle has the same kinetic energy K in the laboratory. The collisions may be viewed as totally inelastic, in that the rest energy of the two colliding protons, plus all the available kinetic energy, can be used to generate new particles and to endow them with kinetic energy. Show that, in contrast to the previous problem, the available energy in this arrangement can be written in the form

$$\mathcal{E} = 2mc^2\left(1 + \frac{K}{2mc^2}\right).$$

`Four-momenum before the collision:`

$$P = m\gamma c(1, \beta) + m\gamma c(1, -\beta) = 2m\gamma c(1, 0)$$

`After the collision the four-momenum is` P' `with` $P'^2 = M^2c^2$`. But we know that` $P^2 = P'^2$`:`

$$M^2c^2 = 4m^2\gamma^2c^2$$

`>From the relation`

$$K = (\gamma - 1)mc^2$$

`you can find` γ `as a function of` K`:`

$$\gamma = 1 + \frac{K}{mc^2}$$

$$\Rightarrow \mathcal{E} = Mc^2 = 2m\gamma c^2 = 2mc^2(1 + \frac{K}{mc^2})$$

(b) How much energy is made available when 100-GeV protons are used in this fashion?

`for` $K = 100\ GeV$`,`

$$\mathcal{E} = 2(0.938\ GeV)(1 + 100/0.938) = 202\ GeV$$

(c) What proton energy would be required to make 100 GeV available?

`For` $\mathcal{E} = 100\ GeV$`,`

$$100\ GeV = 2(0.938\ GeV)(1 + K/0.938)$$

$$K = 43.0\ GeV$$

(d) Compare these results with those of the previous problem and comment on the advantages of storage rings.

`This means that head-on collision has more advantages because we need less Kinetic energy to produce a given mass or with a given Kinetic energy we can produce more massive particles.`

17. Compton scattering (5 points)

Following the discussion in lecture, consider Compton scattering, in which a photon with wavelength λ_0 scatters from an electron at rest. After the collision, a photon of wavelength λ is emitted at an angle θ relative to the direction of the initial photon and the electron recoils with the velocity and direction required by momentum and energy conservation. Derive the relationship between the scattering angle θ and the wavelength change $\Delta\lambda = \lambda = \lambda_0$ for Compton scattering:

$$\Delta\lambda = \frac{h}{mc}(1 - \cos\theta)$$

(a) Use invariants as in lecture.

`A photon with four-momentum` $p_\gamma = (E_\gamma, E_\gamma, 0, 0)$ `strikes a stationary electron with` $p_i = (m_e, 0, 0, 0)$`. We have temporarily suppressed factors of` c`. The photon is scattered to a four-momentum` $p_{\gamma'} = (E_{\gamma'}, E_{\gamma'}\cos\theta, E_{\gamma'}\sin\theta, 0)$ `where` θ `denotes the angle that the scattered photon makes with the direction of the incident photon. The electron is scattered to some four-momentum` p_f`.`

$$p_\gamma + p_i = p_{\gamma'} + p_f$$

$$\Rightarrow p_f^2 = (p_\gamma + p_i - p_{\gamma'})^2$$

$$\Rightarrow m_e^2 = m_e^2 + 2m_e(E_\gamma - E_{\gamma'}) - 2E_\gamma E_{\gamma'}(1 - \cos\theta)$$

$$\Rightarrow E_{\gamma'} = \frac{m_e c^2 E_\gamma}{m_e c^2 + (1-\cos\theta)E_\gamma} \quad \texttt{(with factors of c restored).}$$

`Now the initial and final wavelengths (`λ `and` λ'`) are obtained from`

$$E_\gamma = \frac{hc}{\lambda}\,, E_{\gamma'} = \frac{hc}{\lambda'}\,.$$

`So,`

$$\lambda' = \lambda + \frac{h}{mc}(1-\cos\theta)$$

$$\Rightarrow \Delta\lambda = \frac{h}{mc}(1-\cos\theta)$$

`where` $\Delta\lambda$ `is the final wavelength minus the initial wavelength.`

(b) Use energy conservation, momentum conservation in the x-direction, and momentum conservation in the y-direction, and eliminate the appropriate variables.

`Let four momentum of final electron be denoted by` $(E'_e, p'\cos\phi, -p'\sin\phi, 0)$`.`
`Energy conservation gives`

$$\begin{aligned} E_\gamma + m_e &= E'_\gamma + E'_e \\ \Rightarrow \qquad E'_e &= E_\gamma + m_e - E'_\gamma \end{aligned}$$

`Momentum conservation along x-direction gives`

$$\begin{aligned} E_\gamma &= E'_\gamma\cos\theta + p'\cos\phi \\ \Rightarrow \quad p'\cos\phi &= E_\gamma - E'_\gamma\cos\theta \end{aligned}$$

`and along y-direction`

$$p'\sin\phi = E'_\gamma\sin\theta\,.$$

`Squaring and adding last two equation we get`

$$(p')^2 = E_\gamma^2 + (E'_\gamma)^2 - 2E_\gamma E'_\gamma\cos\theta\,.$$

`Subtracting the last equation from square of energy conservation equation gives`

$$\begin{aligned} (E'_e)^2 - (p')^2 &= m_e^2 + 2m_e(E_\gamma - E'_\gamma) - 2E_\gamma E'_\gamma(1-\cos\theta) \\ \Rightarrow \qquad m_e(E_\gamma - E'_\gamma) &= E_\gamma E'_\gamma(1-\cos\theta) \\ \Rightarrow \qquad \left(\frac{1}{E'_\gamma} - \frac{1}{E_\gamma}\right) &= \frac{1}{m_e}(1-\cos\theta) \\ \Rightarrow \qquad \lambda' - \lambda &= \frac{h}{m_e c}(1-\cos\theta) \qquad \texttt{(substituting factors of } c \texttt{ back)} \\ \Rightarrow \qquad \Delta\lambda &= \frac{h}{m_e c}(1-\cos\theta). \end{aligned}$$

18. Production of hyperons at rest (7 points)

The K^- meson and the Λ^0 hyperon are two commonly encountered unstable particles. For example, they are commonly produced in air showers by cosmic rays and several of each have whizzed through you during the time you have been working on this problem set. The reaction

$$K^- + p \to \Lambda^0 + \pi^0$$

can be used to make Λ^0's at rest in the laboratory by scattering K^- mesons off a stationary proton (hydrogen) target.

The rest energies of all the particles involed are: $m_p c^2 = 939$ MeV, $m_{K^\pm} c^2 = 494$ MeV, $m_{\pi^0} c^2 = 135$ MeV, $m_{\Lambda^0} c^2 = 1116$ MeV.

(a) Find the energy of the incident K^- beam required to just produce Λ^0 hyperons at rest in the lab.

Let $p_K, p_p, p_\Lambda, p_\pi$ be the four-momenta of the K^-, p, Λ^0, π^0 particles respectively. We know that

$$p_K = (E_K, |\vec{p}_K|)\,,$$
$$p_p = (m_p, 0)\,,$$
$$p_\Lambda = (m_\Lambda, 0)\,.$$

>From conservation of four-momentum,

$$p_K + p_p = p_\Lambda + p_\pi$$
$$\Rightarrow p_\pi^2 = (p_K + p_p - p_\Lambda)^2$$
$$\Rightarrow m_\pi^2 = m_K^2 + m_p^2 + m_\Lambda^2 + 2(m_p - m_\Lambda)E_K - 2m_p m_\Lambda$$
$$\Rightarrow E_K = \frac{m_K^2 - m_\pi^2 + (m_\Lambda - m_p)^2}{2(m_\Lambda - m_p)} \approx \boxed{726 \text{ MeV}}\,.$$

(b) What is the π^0 energy for this "magic" K^- energy?

$$E_\pi = E_K + E_p - E_\Lambda = E_K + m_p - m_\Lambda \approx \boxed{549 \text{ MeV}}\,.$$

(c) Check momentum conservation.

$$|\vec{p}_K| = \sqrt{E_K^2 - m_K^2} \approx 533 MeV\,,$$
$$|\vec{p}_\pi| = \sqrt{E_\pi^2 - m_\pi^2} \approx 533 MeV\,.$$

The hyperon and proton have no 3-momentum. Hence, momentum is conserved as long as the K and the π are directed in the same direction.

(d) Could the process be run the other way? That is, could a π^0 beam (assuming one was available) be used to make a K^+ at rest by the reaction $\pi^0 + p \to \Lambda^0 + K^+$?

`If a beam of pions strikes stationary protons and makes kaons at rest, then all the results of the previous calculation will hold with the following substitutions:`

$$K \to \pi\,, \Lambda \to K\,, \pi \to \Lambda\,.$$

`So,`

$$E_\pi == \frac{m_\pi^2 - m_\Lambda^2 + (m_K - m_p)^2}{2(m_K - m_p)} \approx 1156 \ \texttt{MeV} > m_\pi\,,$$

$$E_\Lambda = E_\pi + E_p - E_K = E_\pi + m_p - m_K \approx 1601 \ \texttt{MeV} > m_\Lambda\,.$$

`Thus, the process can be run the other way.`

19. Invariant Product (5 points)

`Lorentz transformation for` A' `and` B' `are given by`

$$\begin{aligned} a_0' &= a_0 \cosh\eta - a_x \sinh\eta \qquad a_x' = a_x \cosh\eta - a_0 \sinh\eta \qquad a_{y,z}' = a_{y,z} \\ b_0' &= b_0 \cosh\eta - b_x \sinh\eta \qquad b_x' = b_x \cosh\eta - b_0 \sinh\eta \qquad b_{y,z}' = b_{y,z} \end{aligned}$$

`Let us explicitly evaluate the dot product and check the invariance. By definition`

$$A' \cdot B' = a_0' b_0' - a_x' b_x' - a_y' b_y' - a_z' b_z'\,.$$

`Using the transformation we get`

$$\begin{aligned} A' \cdot B' &= (a_0 \cosh\eta - a_x \sinh\eta)(b_0 \cosh\eta - b_x \sinh\eta) \\ &\quad -(a_x \cosh\eta - a_0 \sinh\eta)(b_x \cosh\eta - b_0 \sinh\eta) - a_y b_y - a_z b_z \\ &= a_0 b_0 \cosh^2\eta + a_x b_x \sinh^2\eta - (a_0 b_x + a_x b_0)\cosh\eta \sinh\eta \\ &\quad -a_x b_x \cosh^2\eta - a_0 b_0 \sinh^2\eta + (a_x b_0 + a_0 b_x)\cosh\eta \sinh\eta - a_y b_y - a_z b_z \\ &= a_0 b_0 (\cosh^2\eta - \sinh^2\eta) - a_x b_x (\cosh^2\eta - \sinh^2\eta) - a_y b_y - a_z b_z \\ &= a_0 b_0 - a_x b_x - a_y b_y - a_z b_z \\ &= A \cdot B \end{aligned}$$

`where in the second last step we have used identity` $\cosh^2\eta - \sinh^2\eta = 1$. `This verifies the invariance of the product.`