PROBLEM 1: A CIRCLE IN A NON-EUCLIDEAN GEOMETRY

(a) (5 points) The metric is given by

\[ ds^2 = a^2 \left\{ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right\} . \]

The metric allows us to find the proper, or physical, distance associated with a given coordinate displacement. For our problem, we are interested in the physical distance that results from motion in the \( \phi \) direction, with the other coordinates held fixed, as shown at the right. For an infinitesimal segment of the circle, \( r = r_0 \) and \( \theta = \pi/2 \) are held constant; thus, \( dr = d\theta = 0 \), while \( d\phi \) is arbitrary. Plugging these values into the metric, we find that the physical arc length for an infinitesimal piece \( d\phi \) of the circumference is given by

\[ ds^2 = a^2 \{ r_0^2 d\phi^2 \} \]

and therefore

\[ ds = ar_0 d\phi \ . \]

To find the total circumference, we must integrate \( \phi \) from 0 to \( 2\pi \), so

\[ S = \int ds = ar_0 \int_0^{2\pi} d\phi = \boxed{2\pi ar_0} \ . \]

(b) (5 points) We now want to find the radius of the circle, so we must find the physical path length that corresponds to an infinitesimal displacement of the radial coordinate, with all angles held fixed. With \( \theta = \pi/2 \) and \( \phi = \text{const.} \), \( d\theta = d\phi = 0 \), the metric becomes

\[ ds^2 = a^2 \{ r_0^2 d\phi^2 \} \]
\[ ds^2 = a^2 \left\{ \frac{dr^2}{1 - kr^2} \right\} , \]

and therefore

\[ ds = \frac{adr}{\sqrt{1 - kr^2}} . \]

To find the radius, we simply integrate the infinitesimal displacement,

\[ \rho = \int ds = \int_0^{r_0} \frac{adr}{\sqrt{1 - kr^2}} . \]

For a closed universe, \( k = 1 \) and

\[ \rho = \int_0^{r_0} \frac{adr}{\sqrt{1 - r^2}} = a r \left[ \frac{1}{\sqrt{1 - r^2}} \right]_0^{r_0} = a \sin^{-1} r_0 . \]

For an open universe, \( k = -1 \) and

\[ \rho = \int_0^{r_0} \frac{adr}{\sqrt{1 + r^2}} = a \sinh^{-1} r \left[ \frac{1}{\sqrt{1 + r^2}} \right]_0^{r_0} = a \sinh^{-1} r_0 . \]

(c) (5 points) For \( k = 1 \), \( S = 2\pi ar_0 \) and \( \rho = a \sin^{-1} r_0 \)

\[ \implies S = 2\pi a \sin \left( \frac{\rho}{a} \right) . \]

Note that for \( \rho \ll a \), \( \sin \left( \frac{\rho}{a} \right) \approx \frac{\rho}{a} \) and so \( S \approx 2\pi \rho \), in agreement with Euclidean geometry. For \( \rho > 0 \), \( \sin(\rho/a) < \rho/a \), so \( S < 2\pi \rho \).

For \( k = -1 \), \( S = 2\pi ar_0 \) and \( \rho = a \sinh^{-1} r_0 \)

\[ \implies S = 2\pi a \sinh \left( \frac{\rho}{a} \right) . \]

Again, if \( \rho \ll a \), \( \sinh(\rho/a) \approx \rho/a \) and \( S \approx 2\pi \rho \). For \( \rho > 0 \), \( \sinh(\rho/a) > \rho/a \), so \( S > 2\pi \rho \).
PROBLEM 2: VOLUME OF A CLOSED UNIVERSE (15 points)

The metric for the closed universe can be written as

$$ds^2 = R^2 \left[ d\psi^2 + \sin^2 \psi \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right] ,$$

which is Eq. (5.14). For comparison, the metric for the surface of a sphere of radius $R$ is given by Eq. (5.8),

$$ds^2 = R^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) .$$

By comparing these two, one sees that the set of points described by $\psi$ = constant (varying $\theta$ and $\phi$) has the same metric as a sphere of radius $r = R \sin \psi$. We can save ourselves some trouble in calculating by remembering that the area of such a spherical surface of radius $r$ is $4\pi r^2 = 4\pi R^2 \sin^2 \psi$.

The volume of the spherical shell shown in the problem is just the area times the thickness. The thickness is not $d\psi$, since $\psi$ is only a coordinate — remember that in curved space a coordinate and a distance are two different things. The distance is given by the metric. Consider in this case a radial line extending from $\psi$ to $\psi + d\psi$, at constant $\theta$ and $\phi$. Then

$$ds^2 = R^2 d\psi^2 ,$$

and so the length of the line segment is $ds = R d\psi$.

The volume of the spherical shell is then given by

$$dV = \left[ 4\pi R^2 \sin^2 \psi \right] R d\psi .$$

We must now integrate over the range of $\psi$. The variable $\psi$ was introduced in Figure 5.10 and Eqs. (5.12), and it was defined as the angle between the radial line and the positive $w$-axis. This angle ranges from 0, when the two lines are parallel, to $\pi$ when they point in opposite directions. So

$$V = 4\pi R^3 \int_0^{\pi} \sin^2 \psi \, d\psi .$$

Integrating,

$$\int_0^{\pi} \sin^2 \psi \, d\psi = \int_0^{\pi} \frac{1 - \cos 2\psi}{2} d\psi$$

$$= \frac{1}{2} \left\{ \left[ \psi \right]_0^{\pi} - \frac{1}{2} \sin 2\psi \right\}_0^{\pi}$$

$$= \frac{\pi}{2} ,$$
where the $\sin 2\psi$ term gives no contribution, since it vanishes at both endpoints. So

$$V = 2\pi^2 R^3.$$ 

Alternatively, there is a famous (and very useful) mathematical “trick” for integrating $\sin^2 \psi$ over any interval which is a multiple of $\pi/2$. Over such an interval $\sin^2 \psi$ varies over its full range, and $\cos^2 \psi$ would do the same (although out of phase). Using the fact that $\cos^2 \psi + \sin^2 \psi = 1$, it follows that $\cos^2 \psi$ and $\sin^2 \psi$ must each average to $1/2$ when integrated over any interval which is a multiple of $\pi/2$. The integral

$$\int_0^\pi \sin^2 \psi d\psi = \int_0^\pi d\psi \times [\text{average of } \sin^2 \theta] = \frac{\pi}{2}.$$ 

Another way of expressing the answer is to use Eq. (5.21),

$$R^2(t) = \frac{a^2(t)}{k},$$

to eliminate $R$ in favor of $a$, so

$$V = \frac{2\pi^2 a^3(t)}{k^{3/2}}.$$ 

**PROBLEM 3: SURFACE BRIGHTNESS IN A CLOSED UNIVERSE**

*(25 points)*

In this problem we use the form of the metric

$$ds^2 = -c^2 dt^2 + a^2(t)\left\{d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta \, d\phi^2)\right\}$$

(a) *(10 points)* Following the hint, we draw Robertson-Walker coordinates with the galaxy $G$ in the center. The radial coordinate of the detector on Earth will be

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* Solution by Barton Zwiebach, based on a prior version by Alan Guth.*
The diagram also shows a sphere at the same radial coordinate $\psi_G$:

Since the speed of light is independent of angle, all the photons that left the galaxy $G$ at time $t_G$ are arriving at the $\psi = \psi_G$ sphere at the present time $t_0$. To calculate the power received by the detector we need to know what fraction of those photons hit the detector. The fraction is simply the area $A$ of the detector divided by the area $A_s(t_0)$ of the sphere at time $t_0$. The area of the sphere can be calculated by restricting the metric to the case $dt = d\psi = 0$, $t = t_0$, $\psi = \psi_G$:

$$ds^2 = a^2(t_0) \sin^2 \psi_G (d\theta^2 + \sin^2 \theta \, d\phi^2).$$

This expression is identical to the metric of the surface of a sphere of radius $r = a(t_0) \sin \psi_G$. The area is therefore $A_s(t_0) = 4\pi r^2 = 4\pi a^2(t_0) \sin^2 \psi_G$. So,

$$\text{fraction} = \frac{\text{area of detector}}{\text{area of sphere}} = \frac{A}{4\pi a^2(t_0) \sin^2 \psi_G}.$$  

The power hitting the detector is further reduced by one factor of $(1 + z) = a(t_0)/a(t_G)$ because the frequency, and hence the energy, of each photon is reduced by this factor. In addition, the power is reduced by another factor of $(1 + z)$ because the rate of arrival of photons is reduced by this factor. Thus, if $P$ is the power that the galaxy was emitting at time $t_G$, then the power received by the detector today is

$$P_{\text{received}} = P \frac{A}{4\pi a^2(t_0) \sin^2 \psi_G} \left[\frac{a(t_G)}{a(t_0)}\right]^2$$

$$= P \frac{A a^2(t_G)}{4\pi a^4(t_0) \sin^2 \psi_G}.$$
The flux is given by

\[
J = \frac{P_{\text{received}}}{A} = \frac{P a^2(t_G)}{4\pi a^4(t_0) \sin^2 \psi_G}.
\]

(b) (10 points) If we choose the axis shown vertically in the diagram to be the z-axis, then the angle labeled $\Delta \theta$ will represent an increment of the Robertson-Walker coordinate $\theta$, as the label $\Delta \theta$ suggests:

At time $t_G$ the distance between the two edges of the galaxy is given, according to the metric, by

\[
ds = a(t_G) \sin \psi_G \Delta \theta,
\]

where I have assumed that $\Delta \theta \ll 1$. But the problem tells us that this distance is $w$, so

\[
w = a(t_G) \sin \psi_G \Delta \theta \implies \Delta \theta = \frac{w}{a(t_G) \sin \psi_G}.
\]

Remarks. In solving the problem this way, we used the diagram above only to label the coordinates, but we needed the metric to determine the angle. It would have been incorrect to assume that $\Delta \theta$ is calculated by dividing a comoving distance $w/a(t_G)$ by the radial distance $\psi_G$. The problem is that in curved space the angle between two geodesics cannot be calculated by dividing an arc length over a radial distance. Think, for example, of two geodesics starting at the north pole of planet Earth at an angle of 90$^\circ$. By the time they reach the equator, the distance between them, along the equator, is equal to the radial distance from the north pole to the equator. A naive calculation of the angle would then give 1 radian, smaller than the correct angle of 90$^\circ$. 
By taking advantage of the fact that a closed universe can be viewed as a sphere embedded in a Euclidean space, the two dimensional analog of the problem can be visualized nicely. As shown in the figure to the right, we assume planet Earth is at the north pole $P$ and a galaxy, shown as a small disk, is on a latitude circle with some value of $\theta$. We also show two geodesics (or light ray trajectories) that leave the two sides $A$ and $B$ of the galaxy and reach $P$. Suppose the distance between the sides $A$ and $B$ of the galaxy is $w$, what is the angle $\Delta \phi$ at $P$? The angle is not $w$ divided by the distance $a\theta$ from $P$ to the galaxy. To find the correct value we must first show that the angle $\Delta \phi$ or, equivalently, the angle $APB$ is equal to the angle $AQB$.

One way to see this is to note that the vertical projection of the line $PA$ to the plane through the latitude circle gives the line $QA$ and the vertical projection of the line $PB$ gives the line $QB$. In a very small neighborhood of $P$ the surface of the sphere is approximately flat and locally parallel to the plane through the latitude circle. The projection then does pretty much nothing, showing that the angles are the same. Note that this means that the angle at $P$ can be calculated by dividing arc length by radial distance only in the limit as the radial distance goes to zero.

To calculate $\Delta \phi$ we use the disk through the latitude circle. On this circle the galaxy spans a distance $w$ and the radial distance is equal to the radius $a \sin \theta$. Since this is a circle on flat space, we find

$$\Delta \phi = \frac{w}{a \sin \theta}.$$ 

By our argument above this is the correct value for the angle seen at $P$. The more operational way to obtain this answer uses the metric on the sphere:

$$ds^2 = a^2(d\theta^2 + \sin^2 \theta d\phi^2).$$

The galaxy corresponds to $d\theta = 0$ and $d\phi = \Delta \phi$. We then have

$$w = ds = a \sin \theta \Delta \phi,$$

which coincides with the answer obtained before.

(c) (5 points) To evaluate the solid angle subtended by the galaxy, imagine surrounding the observer by a small sphere of arbitrary radius $r$. The galaxy would
appear on this sphere as a disk with an angular radius $\Delta \theta/2$, which implies a radius of $r \Delta \theta/2$, and an area $A = \pi r^2 \Delta \theta^2/4$. The solid angle is given by

$$\Delta \Omega \equiv A/r^2 = \frac{\pi \Delta \theta^2}{4}.$$ 

Using the answers from the previous two parts, the surface brightness is given by

$$\sigma = \frac{J}{\Delta \Omega} = \frac{4J}{\pi \Delta \theta^2} = \frac{4Ja^2(t_G) \sin^2 \psi_G}{\pi w^2} = \frac{Pa^4(t_G)}{\pi^2 w^2 a^4(t_0)} = \frac{P}{\pi^2 w^2 (1+z)^4}.$$

While we derived this formula for a closed universe, we would have found the same result in an open or flat universe.

Note that this result implies that for $z \ll 1$, the surface brightness is independent of distance. This result is consistent with Euclidean geometry, which says that both the energy flux and the solid angle are inversely proportional to the square of the distance, so the surface brightness is independent of distance.

Recall that in Problem Set 2 (2016) we discussed the most distant galaxy with a well-determined redshift, with $z = 11.1$. Note that for this galaxy the surface brightness is suppressed by a whopping factor of $(1+z)^4 \approx 21,400$, which indicates why such high redshift objects are difficult to see!

**PROBLEM 4: TRAJECTORIES AND DISTANCES IN AN OPEN UNIVERSE (30 points)**

(a) (5 points) The geodesic is along a radial line, so $d\theta = d\phi = 0$. Then $d\tau = 0$, which is always true for a light pulse traveling in a vacuum, implies that

$$-c^2 dt^2 + a^2(t) d\psi^2 = 0 ,$$

or

$$\frac{d\psi}{dt} = -\frac{c}{a(t)} .$$

Note that Eq. (1) has two roots, $d\psi/dt = \pm c/a(t)$, but the negative sign is right for this problem because the value of $\psi$ for the light pulse starts at $\psi_G$ (which is always positive) and *decreases* to 0. Integrating,

$$d\psi = -\frac{c}{a(t)} dt$$
\[ \int_{\psi_G}^{0} d\psi = - \int_{t_G}^{t_0} \frac{c}{a(t)} \, dt \]

\[ \psi_G = \int_{t_G}^{t_0} \frac{c}{a(t)} \, dt \]

which, since \( \psi_G \) is known, determines \( t_G \) in terms of \( a(t) \).

(b) (5 points) The cosmological red shift is given by

\[ 1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})} . \]

Since \( t_{\text{observed}} = t_0 \) and \( t_{\text{emitted}} = t_G \), it follows that

\[ z_G = \frac{a(t_0)}{a(t_G)} - 1 . \]

(c) (5 points) To find the volume of space with redshifts smaller than that of galaxy \( G \), the first step is to recognize that the redshift increases monotonically with \( \psi_G \). (If you doubt this statement, note that the answer to (a) implies that \( t_G \) decreases monotonically with \( \psi_G \).) Assuming that \( a(t) \) is monotonically increasing, the answer to (b) then implies that \( z_G \) increases monotonically with \( \psi_G \). Thus, the region with \( z \) smaller than that of galaxy \( G \) is the region with \( 0 < \psi < \psi_G \). In other words, we need to find the physical volume of a sphere of radius \( \psi_G \); so conceptually this will be very similar to Problem 2. To integrate the volume of this region, we again divide space into concentric shells, with radial coordinate \( \psi \) and coordinate thickness \( d\psi \), as shown at the right.

The area of the spherical shell is determined by the metric on the surface, which can be obtained from the full metric by treating \( t \) and \( \psi \) as fixed:

\[ ds^2 = a^2(t) \sinh^2 \psi (d\theta^2 + \sin^2 \theta \, d\phi^2) . \]

This expression is identical to the metric of the surface of a sphere of radius \( r = a(t) \sinh \psi \). The area is therefore \( A = 4\pi r^2 = 4\pi a^2(t) \sinh^2 \psi \). Looking
again at the metric, one sees that the physical thickness of the shell is \( ds = a(t) \, d\psi \). The volume of the shell is then

\[
\text{d}V = A a(t) \, d\psi = 4\pi a^3(t) \sinh^2 \psi \, d\psi ,
\]

and the total volume is found by integration:

\[
V = 4\pi a^3(t) \int_0^{\psi_G} \sinh^2 \psi \, d\psi .
\]

**Extension:** You were not asked to evaluate the integral, but it can be done as follows:

\[
\int_0^{\psi_G} \sinh^2 \psi \, d\psi = \int_0^{\psi_G} \left[ \frac{e^\psi - e^{-\psi}}{2} \right]^2 \, d\psi
\]

\[
= \frac{1}{4} \left[ e^{2\psi} + e^{-2\psi} - 2 \right] \left[ \frac{1}{2} e^{2\psi} + \frac{1}{2} e^{-2\psi} - 2 \psi \right] \bigg|_0^{\psi_G}
\]

\[
= \frac{1}{4} \left[ \frac{1}{2} \left( e^{2\psi_G} - e^{-2\psi_G} - 2\psi_G \right) \right]
\]

The volume is then

\[
V = \pi a^3(t) \left[ \sinh(2\psi_G) - 2\psi_G \right].
\]

(d) *5 points* The proper distance between \((\psi, \theta, \phi)\) and \((\psi + d\psi, \theta, \phi)\) is what is measured by a ruler at rest in this coordinate system, but that is exactly the meaning of the \( ds \) that appears in the expression for the metric. Since \( t, \theta, \) and \( \phi \) are constant along the radial line between Earth and the galaxy \( G \), the metric at \( t = t_0 \) reduces to

\[
ds = a(t_0) \, d\psi .
\]

Integrating,

\[
\ell_{\text{prop}} = \int ds = a(t_0) \psi_G .
\]
(e) (5 points) The calculation of the angular size distance is similar to the angular size calculation in Problem 5 of Problem Set 2 (2016), but it is not quite identical. There we were talking about a flat universe, but this time we are interested in a curved universe. The basic method is the same, however, so long as we remember that all distances have to be determined via the metric. Placing the galaxy for convenience along the $x$ axis ($\theta = 0$), we draw it at the time of emission, $t_G$:

We draw the picture at the time of emission, because the photons that we receive today arrive on trajectories that were determined solely by the position of the galaxy at that time. Using the metric, we can express the physical diameter of the galaxy at the time of emission. The only coordinate that changes between the points $A$ and $B$ is $\theta$, so

$$w = ds = a(t_G) \sinh \psi_G \Delta \theta .$$

(Note that we have assumed in the above equation that $\Delta \theta \ll 1$, so that we do not need to distinguish between the angle between $A$ and $B$, which are at opposite ends of the diameter of the sphere, and the angle of visibility, which is bounded by lines which are tangent to the sphere.) The angular size distance is then

$$\ell_{\text{ang}} \equiv \frac{w}{\Delta \theta} = a(t_G) \sinh \psi_G .$$

**Subtlety:** To be sure that the above solution is correct, one must know that the coordinate separation $\Delta \theta$ between the points $A$ and $B$, at the time of emission, is equal to the angular size that we observe today. This equality can be justified by using the fact that we are located at the origin of this coordinate system, and therefore the photons that we detect arrive along radial lines. (One can verify that trajectories that move along radial lines at the speed of light are geodesics, but I will not try to do that here.) The photons that left point $A$,
at \( \theta = \frac{1}{2} \Delta \theta \), will arrive today along the radial line at \( \theta = \frac{1}{2} \Delta \theta \). Similarly, the photons that left point \( B \), at \( \theta = -\frac{1}{2} \Delta \theta \), will arrive today along the radial line at \( \theta = -\frac{1}{2} \Delta \theta \). Thus, the angular size that we observe, the angular separation between these two radial lines, is \( \Delta \theta \).

(f) (5 points) Following the hint, we draw Robertson-Walker coordinates with the galaxy \( G \) in the center. The radial coordinate of the detector, on Earth, will be \( \psi_G \). The diagram also shows a sphere at the same radial coordinate, \( \psi_G \):

Since the speed of light is independent of angle, all the photons that left the galaxy \( G \) at time \( t_G \) are arriving at the \( \psi = \psi_G \) sphere at the present time, \( t_0 \). To calculate the power received by the detector, we need to know what fraction of those photons hit the detector. The fraction is simply the area of the detector divided by the area of the sphere, or

\[
\text{fraction} = \frac{\text{area of detector}}{\text{area of sphere}} = \frac{A}{4\pi a^2(t_0) \sinh^2 \psi_G}.
\]

(The formula for the area was discussed in the answer to (c).) The power hitting the detector is further reduced by one factor of \( (1 + z) = a(t_0)/a(t_G) \) because the frequency, and hence the energy, of each photon is reduced by this factor. In addition, the power is reduced by another factor of \( (1 + z) \) because the rate of arrival of photons is reduced by this factor. Thus, if \( P \) is the power that the galaxy was emitting at time \( t_G \), then the power received by the detector today is

\[
P_{\text{received}} = P \frac{A}{4\pi a^2(t_0) \sinh^2 \psi_G} \left[ \frac{a(t_G)}{a(t_0)} \right]^2
= P \frac{Aa^2(t_G)}{4\pi a^4(t_0) \sinh^2 \psi_G}.
\]
The flux is given by

\[ J = \frac{P_{\text{received}}}{A} = P \frac{a^2(t_G)}{4\pi a^4(t_0) \sinh^2 \psi_G}. \]

From the definition of luminosity distance,

\[ \ell_{\text{lum}} \equiv \sqrt{\frac{P}{4\pi J}} = \frac{a^2(t_0) \sinh \psi_G}{a(t_G)}. \]

Note, by the way, that the luminosity distance and the angular size distance have a simple relationship to each other:

\[ \ell_{\text{lum}} = \left( \frac{a(t_0)}{a(t_G)} \right)^2 \ell_{\text{ang}} = (1 + z)^2 \ell_{\text{ang}}. \]

While we derived this relation for open universes, the same relation would apply in a flat or closed universe. From the answer to part (d), we can see that the proper distance is related in a slightly more complicated way:

\[ \ell_{\text{prop}} = (1 + z) \frac{\psi_G}{\sinh \psi_G} \ell_{\text{ang}} = \frac{1}{1 + z} \frac{\psi_G}{\sinh \psi_G} \ell_{\text{lum}}. \]

The factor \( \psi_G / \sinh \psi_G \) is a geometric factor, independent of the expansion of the universe, which is less than 1 for the open universe case that we are considering. In a closed universe the analogous factor would have been \( \psi_G / \sin \psi_G \), which is greater than 1, and in a flat universe the corresponding factor would be 1.

**PROBLEM 5: THE KLEIN DESCRIPTION OF THE G-B-L GEOMETRY** *(15 points extra credit)*

(a) *(5 points)* The Klein formula for distance is given by

\[ \cosh \left( \frac{d(1, 2)}{a} \right) = \frac{1 - x_1 x_2 - y_1 y_2}{\sqrt{1 - x_1^2 - y_1^2} \sqrt{1 - x_2^2 - y_2^2}}. \]

Defining

\[ x = u \cos \theta \]
\[ y = u \sin \theta \]
for both \((x_1, y_1)\) and \((x_2, y_2)\), one has
\[
x_1 x_2 + y_1 y_2 = u_1 u_2 \cos \theta_1 \cos \theta_2 + u_1 u_2 \sin \theta_1 \sin \theta_2
\]
\[
= u_1 u_2 \left[ \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \right]
\]
\[
= u_1 u_2 \cos (\theta_1 - \theta_2)
\]
\[
\sqrt{1 - x_1^2 - y_1^2} = \sqrt{1 - u_1^2 \cos^2 \theta_1 - u_1^2 \sin^2 \theta_1} = \sqrt{1 - u_1^2}
\]

So
\[
\cos \left[ \frac{d(1, 2)}{a} \right] = \frac{1 - u_1 u_2 \cos (\theta_1 - \theta_2)}{\sqrt{1 - u_1^2} \sqrt{1 - u_2^2}}
\]

(b) \(5\) points As a useful prelude, let us expand the function
\[
f(x) = \frac{1}{\sqrt{b - x}}
\]
in a power series. Note that
\[
f(x) = (b - x)^{-1/2} \quad f(0) = b^{-1/2}
\]
\[
f'(x) = \frac{1}{2} (b - x)^{-3/2} \quad f'(0) = \frac{1}{2} b^{-3/2}
\]
\[
f''(x) = \frac{3}{4} (b - x)^{-5/2} \quad f''(0) = \frac{3}{4} b^{-5/2}
\]
so
\[
f(x) = f(0) + \frac{1}{1!} f'(0) x + \frac{1}{2!} f''(x) x^2 + \ldots
\]
\[
= \frac{1}{\sqrt{b - x}} = \frac{1}{\sqrt{b}} \left\{ 1 + \frac{1}{2} \frac{x}{b} + \frac{3}{8} \frac{x^2}{b^2} + \ldots \right\}
\]

Using
\[
u_1 = u \quad \theta_1 = \theta
\]
\[
u_2 = u + du \quad \theta_2 = \theta + d\theta
\]
\[
d(1, 2) \equiv ds
\]
One has
\[
\cosh \left[ \frac{ds}{a} \right] = 1 + \frac{ds^2}{2!a^2} + \ldots
\]
\[
= \frac{1 - u(u + du) \cos(d\theta)}{\sqrt{1 - u^2} \sqrt{1 - u^2 - 2udu - du^2}}.
\]
\[
= \frac{1}{1 - u^2} \left\{ \left[ 1 - (u^2 + udu) \left( 1 - \frac{1}{2}d\theta^2 + \ldots \right) \right] \times \left[ 1 + \frac{1}{2} \frac{2udu + du^2}{1 - u^2} + \frac{3}{8} \frac{(2udu + du^2)^2}{(1 - u^2)^2} + \ldots \right] \right\}
\]
\[
= \frac{1}{1 - u^2} \left\{ \left[ 1 - u^2 - udu + \frac{1}{2} u^2d\theta^2 + \ldots \right] \times \left[ 1 - u^2 + udu + \frac{1}{2} du^2 + \frac{3}{2} \frac{u^2du^2}{1 - u^2} + \ldots \right] \right\}
\]
\[
= \frac{1}{(1 - u^2)^2} \left\{ (1 - u^2)^2 + (1 - u^2)udu + \frac{1}{2} (1 - u^2)du^2 + \frac{3}{2} u^2du^2 - udu(1 - u^2) - u^2du^2 + \frac{1}{2} u^2d\theta^2(1 - u^2) + \ldots \right\}
\]
\[
= \frac{1}{(1 - u^2)^2} \left\{ (1 - u^2)^2 + \frac{1}{2} du^2 + \frac{1}{2} u^2(1 - u^2)d\theta^2 + \ldots \right\}
\]
\[
= 1 + \frac{1}{2} \frac{du^2}{(1 - u^2)^2} + \frac{1}{2} \frac{u^2d\theta^2}{(1 - u^2)}.
\]
At each stage one can drop all terms higher than second order in the infinitesimals du and d\theta. So
\[
ds^2 = a^2 \left\{ \frac{du^2}{(1 - u^2)^2} + \frac{u^2d\theta^2}{(1 - u^2)} \right\}.
\]
(c) (5 points) The above metric must be compared with the Robertson–Walker form

\[ ds^2 = a^2 \left\{ \frac{dr^2}{1 + r^2} + r^2 d\theta^2 \right\}. \]

Since the coefficients of \( d\theta^2 \) must match, one must have

\[ r^2 = \frac{u^2}{1 - u^2}, \quad \text{or} \quad r = \frac{u}{\sqrt{1 - u^2}}. \]

We must now check to see if the first terms match.

\[
\begin{align*}
    dr &= \frac{du}{\sqrt{1 - u^2}} + \frac{1}{2} \frac{u}{(1 - u^2)^{3/2}} 2udu \\
    &= du \left\{ \frac{1}{\sqrt{1 - u^2}} + \frac{u^2}{(1 - u^2)^{3/2}} \right\} \\
    &= du \left\{ \frac{1}{(1 - u^2)^{3/2}} [1 - u^2 + u^2] \right\} = \frac{du}{(1 - u^2)^{3/2}} \\
    1 + r^2 &= 1 + \frac{u^2}{1 - u^2} = \frac{1}{1 - u^2}
\end{align*}
\]

So,

\[ \frac{dr^2}{1 + r^2} = \frac{du^2}{(1 - u^2)^2}, \quad \text{— It agrees!} \]

Note that \( r \to \infty \) as \( u \to 1 \), so the range of \( u \) is restricted.

**PROBLEM SET 4, PROBLEM 5: ISOTROPY ABOUT TWO POINTS IN EUCLIDEAN SPACES (15 points extra credit)**

The solution to this problem was held over from the previous problem set.

The statement that Weinberg makes, that isotropy about two distinct points implies homogeneity, is true in Euclidean geometry. The proof that Weinberg gives, however, does not cover all cases, because the picture he uses,
might instead look like

where there is no point \( C \) at which the two circles intersect.

If the point \( C \) does not exist, then we must construct a more general argument than the one used by Weinberg. (Of course he was writing a popular-level book, so we should not expect mathematical rigor.) There is no unique way to extend the argument, but one example is based on the following figure:

Suppose that we know that a 2-dimensional Euclidean space is isotropic about two distinct points, which in this diagram are called \( P_1 \) and \( P_2 \). Let \( r_0 > 0 \) be the distance between them. Construct a coordinate system with \( P_1 \) at the origin and \( P_2 \) at \( x = r_0, y = 0 \), as shown in the diagram. In polar coordinates \((r, \theta)\), isotropy about \( P_1 \) implies that no measurable quantity can depend on \( \theta \); i.e., any two points
with the same value of \( r \) are equivalent. So now we have to prove that no measurable quantity can depend on \( r \). Consider a circle of radius \( r_0 \), centered on \( P_2 \), as shown in the diagram. All points on this circle must be equivalent, by isotropy about the point \( P_2 \). By Euclidean geometry, the radial coordinate \( r \) on this circle varies from 0 to \( 2r_0 \), so it follows that all values of \( r \) from 0 to \( 2r_0 \) must be equivalent. The end points are included in this statement; i.e., the radius \( r = 2r_0 \) is equivalent to any smaller value of \( r \). Now consider a circle of radius \( 3r_0 \), centered on \( P_2 \). Again isotropy about \( P_2 \) guarantees that all points on this circle must be equivalent. In Euclidean geometry the radial coordinate \( r \) on this circle varies from \( 2r_0 \) to \( 4r_0 \), so all values of \( r \) from \( 2r_0 \) to \( 4r_0 \) (including the end points) must be equivalent. Thus all values of \( r \) from 0 to \( 4r_0 \) must be equivalent to \( r = 2r_0 \), and hence all values of \( r \) from 0 to \( 4r_0 \) (including the end points) are equivalent. Then one can consider a circle about \( P_2 \) of radius \( r = 5r_0 \), which extends the equivalence to the range 0 to \( 6r_0 \). One can continue indefinitely, proving that all values of \( r \) are equivalent. The argument works in 3 spatial dimensions as well, replacing circles by spheres and \( \theta \) by \((\theta, \phi)\).