PROBLEM SET 7

DUE DATE: Friday, November 4, 2016, at 4:00 pm.

READING ASSIGNMENT: Steven Weinberg, *The First Three Minutes*, Chapter 8 (Epilogue: The Prospect Ahead), and Afterword: Cosmology Since 1977. There is no reading assignment from Ryden this week, but if you would like to read ahead, you will be asked to read Chapter 9 (*The Cosmic Microwave Background*) and Chapter 11 (*Inflation and the Very Early Universe*) before the end of the semester.

UPCOMING QUIZ: Quiz 2, Wednesday, November 9, 2016.

SHORT-TERM CALENDAR:

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PROBLEM 1: GAS PRESSURE AND ENERGY CONSERVATION (25 points)

In this problem we will pursue the implications of the conservation of energy. Consider first a gas contained in a chamber with a movable piston, as shown below:

![Diagram of a gas chamber with a movable piston]
Let $U$ denote the total energy of the gas, and let $p$ denote the pressure. Suppose that the piston is moved a distance $dx$ to the right. (We suppose that the motion is slow, so that the gas particles have time to respond and to maintain a uniform pressure throughout the volume.) The gas exerts a force $pA$ on the piston, so the gas does work $dW = pA dx$ as the piston is moved. Note that the volume increases by an amount $dV = Adx$, so $dW = pdV$. The energy of the gas decreases by this amount, so

$$dU = -pdV . \quad (P1.1)$$

It turns out that this relation is valid whenever the volume of a gas is changed, regardless of the shape of the volume.

Now consider a homogeneous, isotropic, expanding universe, described by a scale factor $a(t)$. Let $u$ denote the energy density of the gas that fills it. (Remember that $u = \rho c^2$, where $\rho$ is the mass density of the gas.) We will consider a fixed coordinate volume $V_{\text{coord}}$, so the physical volume will vary as

$$V_{\text{phys}}(t) = a^3(t) V_{\text{coord}} . \quad (P1.2)$$

The energy of the gas in this region is then given by

$$U = V_{\text{phys}} u . \quad (P1.3)$$

(a) (9 points) Using these relations, show that

$$\frac{d}{dt} \left( a^3 \rho c^2 \right) = -p \frac{d}{dt} (a^3) , \quad (P1.4)$$

and then that

$$\dot{\rho} = -3 \frac{\dot{a}}{a} \left( \rho + \frac{p}{c^2} \right) , \quad (P1.5)$$

where the dot denotes differentiation with respect to $t$.

(b) (8 points) The scale factor evolves according to the relation

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3} G \rho - \frac{kc^2}{a^2} . \quad (P1.6)$$

Using Eqs. (P1.5) and (P1.6), show that

$$\ddot{a} = -\frac{4\pi}{3} G \left( \rho + \frac{3p}{c^2} \right) a . \quad (P1.7)$$

This equation describes directly the deceleration of the cosmic expansion. Note that there are contributions from the mass density $\rho$, but also from the pressure $p$.

(c) (8 points) So far our equations have been valid for any sort of a gas, but let us now specialize to the case of black-body radiation. For this case we know that $\rho = bT^4$, where $b$ is a constant and $T$ is the temperature. We also know that as the universe expands, $aT$ remains constant. Using these facts and Eq. (P1.5), find an expression for $p$ in terms of $\rho$. 
**PROBLEM 2: THE EFFECT OF PRESSURE ON COSMOLOGICAL EVOLUTION (25 points)**

A radiation-dominated universe behaves differently from a matter-dominated universe because the pressure of the radiation is significant. In this problem we explore the role of pressure for several fictitious forms of matter.

(a) *(8 points)* For the first fictitious form of matter, the mass density $\rho$ decreases as the scale factor $a(t)$ grows, with the relation

$$\rho(t) \propto \frac{1}{a^6(t)}.$$  

What is the pressure of this form of matter? [Hint: the answer is proportional to the mass density.]

(b) *(9 points)* Find the behavior of the scale factor $a(t)$ for a flat universe dominated by the form of matter described in part (a). You should be able to determine the function $a(t)$ up to a constant factor.

(c) *(8 points)* Now consider a universe dominated by a different form of fictitious matter, with a pressure given by

$$p = \frac{1}{2} \rho c^2.$$  

As the universe expands, the mass density of this form of matter behaves as

$$\rho(t) \propto \frac{1}{a^n(t)}.$$  

Find the power $n$.

**PROBLEM 3: TIME EVOLUTION OF A UNIVERSE WITH MYSTERIOUS STUFF (15 points)**

Suppose that a model universe is filled with a peculiar form of matter for which

$$\rho \propto \frac{1}{a^5(t)}.$$  

Assuming that the model universe is flat, calculate

(a) *(4 points)* The behavior of the scale factor, $a(t)$. You should be able to find $a(t)$ up to an arbitrary constant of proportionality.

(b) *(3 points)* The value of the Hubble parameter $H(t)$, as a function of $t$.

(c) *(4 points)* The physical horizon distance, $\ell_{p,\text{horizon}}(t)$.

(d) *(4 points)* The mass density $\rho(t)$. 
PROBLEM 4: EFFECT OF AN EXTRA NEUTRINO SPECIES (15 points)

According to the standard assumptions (which were used in the lecture notes), there are three species of effectively massless neutrinos. In the temperature range of $1 \text{ MeV} < kT < 100 \text{ MeV}$, the mass density of the universe is believed to have been dominated by the black-body radiation of photons, electron-positron pairs, and these neutrinos, all of which were in thermal equilibrium.

(a) (5 points) Under these assumptions, how long did it take (starting from the instant of the big bang) for the temperature to fall to the value such that $kT = 1 \text{ MeV}$?

(b) (5 points) How much time would it have taken if there were one other species of massless neutrino, in addition to the three which we are currently assuming?

(c) (5 points) What would be the mass density of the universe when $kT = 1 \text{ MeV}$ under the standard assumptions, and what would it be if there were one other species of massless neutrino?

Total points for Problem Set 7: 80.