(a) From Eqs. (P1.2) and (P1.3) of the problem statement one has

\[ U = V_{\text{phys}}u = a^3(t)V_{\text{coord}}u(t) . \]

If the change described by Eq. (P1.1) happens over a time interval \( dt \), then

\[ dU = -pdV \implies \frac{dU}{dt} = -p\frac{dV}{dt} . \]

Remembering that \( V_{\text{coord}} \) does not vary with time, and using the chain rule for the differentiation of products of functions,

\[ \frac{dU}{dt} = V_{\text{coord}} \frac{d}{dt} (a^3u) = V_{\text{coord}} \frac{d}{dt} (a^3 \rho c^2) \]

and

\[ \frac{dV}{dt} = V_{\text{coord}} \frac{d}{dt} (a^3) . \]

So

\[
\frac{d}{dt} (a^3 \rho c^2) = -p \frac{d}{dt} (a^3) .
\]

Then, using the chain rule again,

\[ 3a^2 \dot{a} \rho c^2 + a^3 \dot{\rho} c^2 = -3p a^2 \dot{a} . \]

Dividing by \( a^3 c^2 \) and rearranging,

\[
\dot{\rho} = -3 \frac{\dot{a}}{a} \left( \rho + \frac{p}{c^2} \right) .
\]

(b) We start by rewriting the Friedmann equation (Eq. (P1.6) of the problem statement),

\[ \dot{a}^2 = \frac{8\pi}{3} G\rho a^2 - k c^2 . \]
Differentiating with respect to time, we have

\[ 2\ddot{a} \dddot{a} = \frac{8\pi}{3} G \dot{\rho} a^2 + \frac{16\pi}{3} G \rho \dot{a} . \]

Using the result of part (a) for \( \dot{\rho} \) and dividing by \( 2\dot{a} \) yields an equation for \( \ddot{a} \),

\[ \ddot{a} = \frac{8\pi}{3} G \left[ -\frac{3}{2} a \left( \rho + \frac{p}{c^2} \right) \right] + \frac{8\pi}{3} G \rho a \implies \]

\[ \ddot{a} = -\frac{4\pi}{3} G \left( \rho + \frac{3p}{c^2} \right) a . \]

**NOTE:** Although we derived Eq. (P1.6) of the problem set in the context of Newtonian cosmology, the same equation holds exactly in the general relativistic treatment of the same problem. The equation above for \( \ddot{a} \) also holds exactly in general relativity.

(c) To make use of the result of part (a), it would be helpful to eliminate \( a \) in favor of \( T \). Using \( aT = \text{const} \), note that

\[ a = \frac{\text{const}}{T} \implies \dot{a} = -\frac{\text{const}}{T^2} \dot{T} = -a \frac{T}{T} , \]

so Eq. (P1.5) becomes

\[ \dot{\rho} = 3 \frac{\dot{T}}{T} \left( \rho + \frac{p}{c^2} \right) . \]

Then using \( \rho = aT^4 \),

\[ \dot{\rho} = 4aT^3 \dot{T} = 3 \frac{\dot{T}}{T} \left( aT^4 + \frac{p}{c^2} \right) . \]

Solving for \( p \) gives

\[ p = \frac{1}{3} aT^4c^2 = \frac{1}{3} \rho c^2 . \]

Note that this is very different from ordinary nonrelativistic gases. For the air in this room, \( p \approx 10^{-12} \rho c^2 \).
PROBLEM 2: THE EFFECT OF PRESSURE ON COSMOLOGICAL EVOLUTION

(a) This problem is answered most easily by starting from the cosmological formula for energy conservation, which I remember most easily in the form motivated by \( dU = -p\, dV \). Using the fact that the energy density \( u \) is equal to \( \rho c^2 \), the energy conservation relation can be written

\[
\frac{dU}{dt} = -p \frac{dV}{dt} \implies \frac{d}{dt} (\rho c^2 a^3) = -p \frac{d}{dt} (a^3) .
\]

Setting \( \rho = \frac{\alpha}{a^6} \) for some constant \( \alpha \), the conservation of energy formula becomes

\[
\frac{d}{dt} \left( \frac{\alpha c^2}{a^3} \right) = -p \frac{d}{dt} (a^3) ,
\]

which implies

\[
-3 \frac{\alpha c^2}{a^4} \frac{da}{dt} = -3p a^2 \frac{da}{dt} .
\]

Thus

\[
p = \frac{\alpha c^2}{a^6} = \frac{\rho c^2}{a^6} .
\]

Alternatively, one may start from the equation for the time derivative of \( \rho \),

\[
\dot{\rho} = -3 \frac{\dot{a}}{a} \left( \rho + \frac{p}{c^2} \right) .
\]

Since \( \rho = \frac{\alpha}{a^6} \), we take the time derivative to find \( \dot{\rho} = -6(\dot{a}/a)\rho \), and therefore

\[
-6 \frac{\dot{a}}{a} \rho = -3 \frac{\dot{a}}{a} \left( \rho + \frac{p}{c^2} \right) ,
\]

and therefore

\[
p = \rho c^2 .
\]

(b) For a flat universe, the Friedmann equation reduces to

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3} G \rho .
\]
Using $\rho \propto 1/a^6$, this implies that

$$\dot{a} = \frac{\beta}{a^2},$$

for some constant $\beta$. Rewriting this as

$$a^2 \, da = \beta \, dt,$$

we can integrate the equation to give

$$\frac{1}{3} a^3 = \beta t + \text{const},$$

where the constant of integration has no effect other than to shift the origin of the time variable $t$. Using the standard big bang convention that $a = 0$ when $t = 0$, the constant of integration vanishes. Thus,

$$a \propto t^{1/3}.$$

The arbitrary constant of proportionality in this answer is consistent with the wording of the problem, which states that “You should be able to determine the function $a(t)$ up to a constant factor.” Note that we could have expressed the constant of proportionality in terms of the constant $\alpha$ that we used in part (a), but there would not really be any point in doing that. The constant $\alpha$ was not a given variable. If the comoving coordinates are measured in “notches,” then $a$ is measured in meters per notch, and the constant of proportionality in our answer can be changed by changing the arbitrary definition of the notch.

(c) We start from the conservation of energy equation in the form

$$\dot{\rho} = -3 \frac{\dot{a}}{a} \left( \rho + \frac{p}{c^2} \right).$$

Substituting $\dot{\rho} = -n(\dot{a}/a)\rho$ and $p = (1/2)\rho c^2$, we have

$$-nH\rho = -3H \left( \frac{3}{2} \rho \right)$$

and therefore

$$n = \frac{9}{2}.$$
PROBLEM 3: TIME EVOLUTION OF A UNIVERSE WITH MYSTERIOUS STUFF (15 points)

(a) (4 points) The Friedmann equation in a flat universe is
\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho.
\]
Substituting \( \rho = \text{const}/a^5 \) and taking the square root of both sides gives
\[
\frac{\dot{a}}{a} = \alpha a^{-5/2},
\]
for some constant \( \alpha \). Rearranging to a form we can integrate,
\[
da a^{3/2} = \alpha dt,
\]
and therefore
\[
\frac{2}{5} a^{5/2} = \alpha t.
\]
Notice that once again we have eliminated the arbitrary integration constant by choosing the Big Bang boundary conditions \( a = 0 \) at \( t = 0 \). Solving for \( a \) yields
\[
a \propto t^{2/5}.
\]

(b) (3 points) The Hubble parameter is, from its definition,
\[
H = \frac{\dot{a}}{a} = \frac{2}{5t},
\]
where we have used the time dependence of \( a(t) \) that we found in part (a). (Notice that we don’t need to know the constant of proportionality left undetermined in part (a), as it cancels between numerator and denominator in this calculation.)

(c) (4 points) Recall that the horizon distance is the physical distance traveled by a light ray since \( t = 0 \),
\[
\ell_{p,\text{horizon}}(t) = a(t) \int_0^t \frac{c dt'}{a(t')}.
\]
Using $a(t) \propto t^{2/5}$, we find

$$\ell_{p, \text{horizon}}(t) = ct^{2/5} \int_0^t dt' t'^{-2/5}$$

or

$$\ell_{p, \text{horizon}}(t) = ct^{2/5} \left( \frac{5}{3} t^{3/5} \right)^{5/3} = \frac{5}{3} ct.$$ 

(d) (4 points) Since we know the Hubble parameter, we can find the mass density $\rho(t)$ easily from the Friedmann equation,

$$\rho(t) = \frac{3H^2}{8\pi G}.$$ 

Using the result from part (b), we find

$$\rho(t) = \frac{3}{50\pi G} \frac{1}{t^2}.$$ 

As a check on our algebra, since we found in (a) that $a \propto t^{2/5}$, and knew at the beginning of the calculation that $\rho \propto a^{-5}$, we should find, as we do here, that $\rho \propto t^{-2}$. Notice, however, that in this case we do not leave our answer in terms of some undetermined constant of proportionality; the units of $\rho$ are not arbitrary, and therefore we care about its normalization.

**PROBLEM 7: EFFECT OF AN EXTRA NEUTRINO SPECIES (15 points)**

Problem held over until the next problem set.