PROBLEM 1: DID YOU DO THE READING? (25 points)

Except for part (d), you should answer these questions by circling the one statement that is correct.

(a) (5 points) In the Epilogue of *The First Three Minutes*, Steve Weinberg wrote: “The more the universe seems comprehensible, the more it also seems pointless.” The sentence was qualified, however, by a closing paragraph that points out that

(i) the quest of the human race to create a better life for all can still give meaning to our lives.

(ii) if the universe cannot give meaning to our lives, then perhaps there is an afterlife that will.

(iii) the complexity and beauty of the laws of physics strongly suggest that the universe must have a purpose, even if we are not aware of what it is.

(iv) the effort to understand the universe gives human life some of the grace of tragedy.

(b) (5 points) In the Afterword of *The First Three Minutes*, Weinberg discusses the baryon number of the universe. (The baryon number of any system is the total number of protons and neutrons (and certain related particles known as hyperons) minus the number of their antiparticles (antiprotons, antineutrons, antihyperons) that are contained in the system.) Weinberg concluded that

(i) baryon number is exactly conserved, so the total baryon number of the universe must be zero. While nuclei in our part of the universe are composed of protons and neutrons, the universe must also contain antimatter regions in which nuclei are composed of antiprotons and antineutrons.

(ii) there appears to be a cosmic excess of matter over antimatter throughout the part of the universe we can observe, and hence a positive density of baryon number. Since baryon number is conserved, this can only be explained by assuming that the excess baryons were put in at the beginning.

(iii) there appears to be a cosmic excess of matter over antimatter throughout the part of the universe we can observe, and hence a positive density of baryon number. This can be taken as a positive hint that baryon number is not conserved, which can happen if there exist as yet undetected heavy “exotic” particles.
(iv) it is possible that baryon number is not exactly conserved, but even if that is the case, it is not possible that the observed excess of matter over antimatter can be explained by the very rare processes that violate baryon number conservation.

Explanation: All students were given credit for this part, whether they answered it correctly or not. I was in San Francisco when I made up this quiz, and due to poor planning I did not have my copy of *The First Three Minutes*. So I found a version online, but I could only find the British version, published by Flamingo/Fontana Paperbacks, rather than the US version published by Basic Books. I assumed that the “Afterword” in the two versions would be the same, but I was wrong! So this question was based on a different “Afterword” than the one that you read. 55% of you still got it right, but obviously the question was not fair. Apologies.

(c) (5 points) In discussing the COBE measurements of the cosmic microwave background, Ryden describes a dipole component of the temperature pattern, for which the temperature of the radiation from one direction is found to be hotter than the temperature of the radiation detected from the opposite direction.

(i) This discovery is important, because it allows us to pinpoint the direction of the point in space where the big bang occurred.

(ii) This is the largest component of the CMB anisotropies, amounting to a 10% variation in the temperature of the radiation.

(iii) In addition to the dipole component, the anisotropies also include contributions from a quadrupole, octupole, etc., all of which are comparable in magnitude.

(iv) This pattern is interpreted as a simple Doppler shift, caused by the net motion of the COBE satellite relative to a frame of reference in which the CMB is almost isotropic.

Explanation: (i) is nonsense, since the conventional big bang theory describes a completely homogeneous universe, which has no single point at which the big bang occurred. (ii) is wrong, because the variations in the temperature of the CMB are much smaller than 10%. The dipole term has a magnitude of about 1/1000 of the mean temperature. (iii) is wrong because the dipole is not comparable to the other terms, because they have magnitudes of only about 1/100,000 of the mean.
(d) \(5 \text{ points}\) (CMB basic facts) Which one of the following statements about CMB is \textit{not} correct:

(i) After the dipole distortion of the CMB is subtracted away, the mean temperature averaging over the sky is \(\langle T \rangle = 2.725 K\).

(ii) After the dipole distortion of the CMB is subtracted away, the root mean square temperature fluctuation is \(\left\langle \left( \frac{\delta T}{T} \right)^2 \right\rangle^{1/2} = 1.1 \times 10^{-3}\).

(iii) The dipole distortion is a simple Doppler shift, caused by the net motion of the observer relative to a frame of reference in which the CMB is isotropic.

(iv) In their groundbreaking paper, Wilson and Penzias reported the measurement of an excess temperature of about 3.5 K that was isotropic, unpolarized, and free from seasonal variations. In a companion paper written by Dicke, Peebles, Roll and Wilkinson, the authors interpreted the radiation to be a relic of an early, hot, dense, and opaque state of the universe.

\textit{Explanation:} The right value is

\[ \left\langle \left( \frac{\delta T}{T} \right)^2 \right\rangle^{1/2} = 1.1 \times 10^{-3}. \]

(e) \(5 \text{ points}\) Inflation is driven by a field that is by definition called the \textit{inflaton} field. In standard inflationary models, the field has the following properties:

(i) The inflaton is a scalar field, and during inflation the energy density of the universe is dominated by its potential energy.

(ii) The inflaton is a vector field, and during inflation the energy density of the universe is dominated by its potential energy.

(iii) The inflaton is a scalar field, and during inflation the energy density of the universe is dominated by its kinetic energy.

(iv) The inflaton is a vector field, and during inflation the energy density of the universe is dominated by its kinetic energy.

(v) The inflaton is a tensor field, which is responsible for only a small fraction of the energy density of the universe during inflation.

\textit{Explanation:} These facts were mentioned in both Section 11.5 (\textit{The Physics of Inflation}) of Ryden’s book, and also in the article that you were asked to read called \textit{Inflation and the New Era of High-Precision Cosmology}, written by me for the Physics Department 2002 newsletter.
PROBLEM 2: THE EFFECT OF PRESSURE ON COSMOLOGICAL EVOLUTION (25 points)

The following problem was Problem 2 of Problem Set 7 (2016), except that some numerical constants have been changed, so the answers will not be identical.

A radiation-dominated universe behaves differently from a matter-dominated universe because the pressure of the radiation is significant. In this problem we explore the role of pressure for several fictitious forms of matter.

(a) (8 points) For the first fictitious form of matter, the mass density $\rho$ decreases as the scale factor $a(t)$ grows, with the relation

$$\rho(t) \propto \frac{1}{a^8(t)} .$$

What is the pressure of this form of matter? [Hint: the answer is proportional to the mass density.]

Answer:

This problem is answered most easily by starting from the cosmological formula for energy conservation, which I remember most easily in the form motivated by $dU = -pdV$. Using the fact that the energy density $u$ is equal to $\rho c^2$, the energy conservation relation can be written

$$\frac{dU}{dt} = -p \frac{dV}{dt} \implies \frac{d}{dt} (\rho c^2 a^3) = -p \frac{d}{dt} (a^3) .$$

Setting

$$\rho = \frac{\alpha}{a^8}$$

for some constant $\alpha$, the conservation of energy formula becomes

$$\frac{d}{dt} \left( \frac{\alpha c^2}{a^5} \right) = -p \frac{d}{dt} (a^3) ,$$

which implies

$$-5 \frac{\alpha c^2}{a^6} \frac{da}{dt} = -3pa^2 \frac{da}{dt} .$$

Thus

$$p = \frac{5 \alpha c^2}{3 a^8} = \frac{5}{3} \rho c^2 .$$
Alternatively, one may start from the equation for the time derivative of $\rho$,

$$\dot{\rho} = -3 \frac{\dot{a}}{a} \left( \rho + \frac{p}{c^2} \right).$$

Since $\rho = \frac{\alpha}{a^8}$, we take the time derivative to find $\dot{\rho} = -8(\dot{a}/a)\rho$, and therefore

$$-8 \frac{\dot{a}}{a} \rho = -3 \frac{\dot{a}}{a} \left( \rho + \frac{p}{c^2} \right),$$

and therefore

$$p = \frac{5}{3} \rho c^2.$$

(b) (9 points) Find the behavior of the scale factor $a(t)$ for a flat universe dominated by the form of matter described in part (a). You should be able to determine the function $a(t)$ up to a constant factor.

**Answer:**

For a flat universe, the Friedmann equation reduces to

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3} G \rho.$$

Using $\rho \propto 1/a^8$, this implies that

$$\dot{a} = \frac{\beta}{a^3},$$

for some constant $\beta$. Rewriting this as

$$a^3 \, da = \beta \, dt,$$

we can integrate the equation to give

$$\frac{1}{4} a^4 = \beta t + \text{const},$$

where the constant of integration has no effect other than to shift the origin of the time variable $t$. Using the standard big bang convention that $a = 0$ when $t = 0$, the constant of integration vanishes. Thus,

$$a \propto t^{1/4}.$$

The arbitrary constant of proportionality in this answer is consistent with the wording of the problem, which states that “You should
be able to determine the function $a(t)$ up to a constant factor.” Note that we could have expressed the constant of proportionality in terms of the constant $\alpha$ that we used in part (a), but there would not really be any point in doing that. The constant $\alpha$ was not a given variable. If the comoving coordinates are measured in “notches,” then $a$ is measured in meters per notch, and the constant of proportionality in our answer can be changed by changing the arbitrary definition of the notch.

(c) (8 points) Now consider a universe dominated by a different form of fictitious matter, with a pressure given by

$$p = \frac{2}{3} \rho c^2 .$$

As the universe expands, the mass density of this form of matter behaves as

$$\rho(t) \propto \frac{1}{a^n(t)} .$$

Find the power $n$.

**Answer:**

We start from the conservation of energy equation in the form

$$\dot{\rho} = -3 \frac{\dot{a}}{a} \left( \rho + \frac{p}{c^2} \right) .$$

Substituting $\dot{\rho} = -n(\dot{a}/a)\rho$ and $p = (2/3)\rho c^2$, we have

$$-nH\rho = -3H \left( \frac{5}{3} \rho \right)$$

and therefore

$$n = 5.$$
PROBLEM 3: THE FREEZE-OUT OF A FICTITIOUS PARTICLE X
(25 points)

Suppose that, in addition to the particles that are known to exist, there also existed a family of three spin-1 particles, $X^+$, $X^-$, and $X^0$, all with masses 0.511 MeV/c$^2$, exactly the same as the electron. The $X^-$ is the antiparticle of the $X^+$, and the $X^0$ is its own antiparticle. Since the $X$’s are spin-1 particles with nonzero mass, each particle has three spin states.

The $X$’s do not interact with neutrinos any more strongly than the electrons and positrons do, so when the $X$’s freeze out, all of their energy and entropy are given to the photons, just like the electron-positron pairs.

(a) (5 points) In thermal equilibrium when $kT \gg 0.511$ MeV/c$^2$, what is the total energy density of the $X^+$, $X^-$, and $X^0$ particles?

Answer:

The formula sheet tells us that the energy density of black-body radiation is

$$u = \frac{g \pi^2}{30} \frac{(kT)^4}{(hc)^3},$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}.$$

Since the $X$ is spin-1, and 1 is an integer, the $X$ particles are bosons and $g = 1$ per spin state. There are 3 species, $X^+$, $X^-$, and $X^0$, and each species we are told has three spin states, so there are a total of 9 spin states, so $g = 9$. Thus,

$$u = 9 \frac{\pi^2}{30} \frac{(kT)^4}{(hc)^3}.$$

Alternatively, one could count the $X^+$ and $X^-$ as one species with a distinct particle and antiparticle, so $g_{X^+X^-}$ is given by

$$g_{X^+X^-} = 1 \times 1 \times 2 \times 3 = 6.$$
The $X^0$ is its own antiparticle, which means that the particle/antiparticle factor is one, so

$$g_{X^0} = 1 \times 1 \times 1 \times 3 = 3,$$

so the total $g$ for $X^+$, $X^-$, and $X^0$ is again equal to 9.

(b) (5 points) In thermal equilibrium when $kT \gg 0.511$ MeV/c$^2$, what is the total number density of the $X^+$, $X^-$, and $X^0$ particles?

Answer:

The formula sheet tells us that the number density of particles in black-body radiation is

$$n = g^* \frac{\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3},$$

where

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions} \end{cases}.$$

For bosons $g^* = g$, so $g^*$ for the $X$ particles is 9. Then

$$n_X = 9 \frac{\zeta(3)}{\pi^2} \frac{(kT)^3}{(\hbar c)^3}.$$

(c) (10 points) The $X$ particles and the electron-positron pairs freeze out of the thermal equilibrium radiation at the same time, as $kT$ decreases from values large compared to 0.511 MeV/c$^2$ to values that are small compared to it. If the $X$'s, electron-positron pairs, photons, and neutrinos were all in thermal equilibrium before this freeze-out, what will be the ratio $T_\nu/T_\gamma$, the ratio of the neutrino temperature to the photon temperature, after the freeze-out?

Answer:

We are told that, when the $X$ particles freeze out, all of their energy and entropy is given to the photons. We use entropy rather than energy to determine the final temperature of the photons, because the entropy in a comoving volume is simply conserved, while the energy density varies as

$$\dot{\rho} = -3 \frac{\dot{a}}{a} \left(\rho + \frac{p}{c^2}\right).$$
Thus, to track the energy, we need to know exactly how $p$ behaves, and the behavior of $p$ during freeze-out is complicated, and we have not calculated it in this course.

The formula sheet tells us that the entropy density of a constituent of black-body radiation is given by

$$s = g \frac{2\pi^2}{45} \frac{k^4 T^3}{(hc)^3}.$$  

If we consider some fixed coordinate volume $V_{\text{coord}}$, the corresponding physical volume is $V_{\text{phys}} = V_{\text{coord}} a^3(t)$, where $a(t)$ is the scale factor. The total entropy of neutrinos in $V_{\text{coord}}$ is then

$$S_{\nu} = g_{\nu} \frac{2\pi^2}{45} \frac{k^4 T^3_{\nu}(t)}{(hc)^3} V_{\text{coord}} a^3(t).$$

The quantities $T_{\nu}(t)$ and $a(t)$ depend on time, but the expression on the right-hand-side does not, since entropy is conserved. For brevity I will write

$$S_{\nu} = g_{\nu} A(t) T^3_{\nu}(t), \quad (1)$$

where

$$A(t) \equiv \frac{2\pi^2}{45} \frac{k^4}{(hc)^3} V_{\text{coord}} a^3(t).$$

The $e^+e^-$ pairs and the $X$'s contribute to the black-body radiation only before the freeze-out, when $kT \gg 0.511 \text{ MeV}/c^2$. Let $t_b$ denote any time before the freeze-out. Before the freeze-out, the total entropy of photons, $e^+e^-$ pairs, and $X$ particles is given by

$$S_{\text{before, } \gamma e X} = (g_{\gamma} + g_{e^+e^-} + g_X) A(t_b) T^3_{\gamma}(t_b).$$  

(2)

I can call the temperature $T_{\gamma}$, because the $e^+e^-$ pairs and the $X$'s (as well as the neutrinos) are all in thermal equilibrium at this point, so they all have the same temperature.

Using $t_a$ to denote an arbitrary time after the freeze-out, the entropy of the photons during this time period can be written

$$S_{\text{after, } \gamma} = g_{\gamma} A(t_a) T^3_{\gamma}(t_a).$$  

(3)

But since the $e^+e^-$ pairs and $X$ particles give all their entropy to the photons, we have

$$S_{\text{after, } \gamma} = S_{\text{before, } \gamma e X}. \quad (4)$$
Then using Eqs. (2) and (3) we find
\[ g_\gamma A(t_a) T_\gamma^3(t_a) = (g_\gamma + g_{e^+e^-} + g_X) A(t_b) T_\gamma^3(t_b). \] (5)

We can rewrite the last factor in Eq. (5) by remembering that Eq. (1) holds at all times, and that \( T_\nu(t_b) = T_\gamma(t_b) \). So,
\[ A(t_b) T_\gamma^3(t_b) = A(t_b) T_\nu^3(t_b) = \frac{S_\nu}{g_\nu} = A(t_a) T_\nu^3(t_a). \] (6)

Substituting Eq. (6) into Eq. (5), we have
\[ g_\gamma A(t_a) T_\gamma^3(t_a) = (g_\gamma + g_{e^+e^-} + g_X) A(t_a) T_\nu^3(t_a), \]
from which we see that
\[ T_\gamma^3(t_a) = \frac{g_\gamma + g_{e^+e^-} + g_X}{g_\gamma} T_\nu^3(t_a), \]
and therefore
\[
\frac{T_\nu(t_a)}{T_\gamma(t_a)} = \left( \frac{g_\gamma}{g_\gamma + g_{e^+e^-} + g_X} \right)^{1/3} = \left( \frac{2}{2 + \frac{7}{2} + 9} \right)^{1/3} = \left( \frac{4}{29} \right)^{1/3}.
\]

(d) (5 points) If the mass of the \( X \)'s was, for example, 0.100 MeV/c\(^2\), so that the electron-positron pairs froze out first, and then the \( X \)'s froze out, would the final ratio \( T_\nu/T_\gamma \) be higher, lower, or the same as the answer to part (c)? Explain your answer in a sentence or two.

**Answer:**

The answer would be the same, since it was completely determined by the conservation equation, Eq. (4) in the above answer. Regardless of the order in which the freeze-outs occurred, the total entropy from the \( e^+e^- \) pairs and the \( X \)'s would ultimately be given to the photons, so the amount of heating of the photons would be the same.
PROBLEM 4: THE TIME $t_d$ OF DECOUPLING (25 points)

The process by which the photons of the cosmic microwave background stop scattering and begin to travel on straight lines is called decoupling, and it happens at a photon temperature of about $T_d \approx 3,000$ K. In Lecture Notes 6 we estimated the time $t_d$ of decoupling, working in the approximation that the universe has been matter-dominated from that time to the present. We found a value of 370,000 years. In this problem we will remove this approximation, although we will not carry out the numerical evaluation needed to compare with the previous answer.

(a) (5 points) Let us define

$$x(t) \equiv \frac{a(t)}{a(t_0)} ,$$

as on the formula sheets, where $t_0$ is the present time. What is the value of $x_d \equiv x(t_d)$? Assume that the entropy of photons is conserved from time $t_d$ to the present, and let $T_0$ denote the present photon temperature.

**Answer:**

If the entropy of photons is conserved, then the entropy density falls as

$$s \propto \frac{1}{a^3(t)} .$$

Since $s \propto T^3$, it follows that

$$T \propto \frac{1}{a(t)} .$$

Thus, the ratio of the scale factors is equal to the inverse of the ratio temperatures:

$$x_d = \frac{T_0}{T_d} .$$

(b) (5 points) Assume that the universe is flat, and that $\Omega_{m,0}$, $\Omega_{\text{rad},0}$, and $\Omega_{\text{vac},0}$ denote the present contributions to $\Omega$ from nonrelativistic matter, radiation, and vacuum energy, respectively. Let $H_0$ denote the present value of the Hubble expansion rate. Write an expression in terms of these quantities for $dx/dt$, the derivative of $x$ with respect to $t$. **Hint:** you may use formulas from the formula sheet without derivation, so this problem should require essentially no work. To receive full credit, your answer should include only terms that make a nonzero contribution to the answer.
**Answer:**

The formula sheet reminds us that

\[
x \frac{dx}{dt} = H_0 \sqrt{\Omega_{m,0} x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0} x^4 + \Omega_{k,0} x^2},
\]

where

\[
\Omega_{k,0} \equiv -\frac{kc^2}{a^2(t_0)H_0^2} = 1 - \Omega_{m,0} - \Omega_{\text{rad},0} - \Omega_{\text{vac},0}.
\]

So for a flat universe \( \Omega_{k,0} = 0 \), and we have

\[
x \frac{dx}{dt} = \frac{H_0}{x} \sqrt{\Omega_{m,0} x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0} x^4}.
\]

(c) (5 points) Write an expression for \( t_d \). If your answer involves an integral, you need not try to evaluate it, but you should be sure that the limits of integration are clearly shown.

**Answer:**

The answer to part (b) can be rewritten as

\[
dt = \frac{x \, dx}{H_0 \sqrt{\Omega_{m,0} x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0} x^4}}.
\]

\( t_d \) is the time that elapses from when the universe has \( x = 0 \) to when it has \( x = x_d \), so

\[
t_d = \frac{1}{H_0} \int_0^{x_d} \frac{x \, dx}{\sqrt{\Omega_{m,0} x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0} x^4}}.
\]

You were of course not asked to evaluate this integral numerically, but we will do that now. We take \( T_0 = 2.7255 \) K from Fixsen et al. (cited in Lecture Notes 6) and the Planck 2015 best fit values of \( H_0 = 67.7 \) km-s\(^{-1}\)-Mpc\(^{-1} \), \( \Omega_{m,0} = 0.309 \), \( \Omega_{\text{vac},0} = 0.691 \). The energy density of radiation (photons plus neutrinos) can then be calculated to give \( \Omega_{\text{rad},0} = 9.2 \times 10^{-5} \) (see Eq. (6.23) of Lecture Notes 6 and the text of the 2nd paragraph of p. 12 of Lecture Notes 7). To keep our model universe exactly flat, I am modifying \( \Omega_{\text{vac},0} \) to set it equal
to $0.691 - \Omega_{\text{rad},0}$, which is well within the uncertainties. Numerical integration then gives 366,000 years, very close to our original estimate. Of course this number is still approximate, since we started with $T_d \approx 3000$ K. In any case, the decoupling of the photons in the CMB is actually a gradual process. In 2003 I modified a standard program called CMBFast to calculate the probability distribution of the time of last scattering (published in https://arxiv.org/abs/astro-ph/0306275), with the following results:

![Diagram](image.png)

The parameters used were $\Omega_{\text{vac},0} = 0.70$, $\Omega_{m,0} = 0.30$, $H_0 = 68 \text{ km}\cdot\text{s}^{-1}\cdot\text{Mpc}^{-1}$. The peak of the curve is at 367,000 years, and the median is at 388,000 years.

(d) (10 points) Now suppose that in addition to the constituents described in part (b), the universe also contains some of the fictitious material from part (a) of Problem 2, with

$$\rho(t) \propto \frac{1}{a^8(t)} .$$

Denote the present contribution to $\Omega$ from this fictitious material as $\Omega_{f,0}$. The universe is still assumed to be flat, so the numerical values of $\Omega_{m,0}$, $\Omega_{\text{rad},0}$, and $\Omega_{\text{vac},0}$ must sum to a smaller value than in parts (b) and (c). With this extra contribution to the mass density of the universe, what is the new expression for $t_d$?

**Answer:**

The derivation starts with the first-order Friedmann equation. Since we are describing a flat universe, we can start with the Friedmann equation for a flat universe,

$$H^2 = \frac{8\pi}{3} G \rho .$$
Now we use the facts that $\rho_m \propto 1/a^3$, $\rho_{\text{rad}} \propto 1/a^4$, $\rho_{\text{vac}} \propto 1$, and $\rho_f \propto 1/a^8$ to write

$$H^2 = \frac{8\pi}{3} G \left[ \frac{\rho_{m,0}}{x^3} + \frac{\rho_{\text{rad},0}}{x^4} + \rho_{\text{vac},0} + \frac{\rho_{f,0}}{x^8} \right].$$

Then we use

$$\rho_{m,0} = \rho_c \Omega_{m,0} = \frac{3H_0^2}{8\pi G} \Omega_{m,0},$$

with similar relations for the other components of the mass density, to rewrite the Friedmann equation as

$$H^2 = H_0^2 \left[ \frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{\text{rad},0}}{x^4} + \Omega_{\text{vac},0} + \frac{\Omega_{f,0}}{x^8} \right].$$

Next we rewrite $H^2$ as

$$H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \left( \frac{\dot{x}}{x} \right)^2,$$

so

$$\left( \frac{\dot{x}}{x} \right)^2 = H_0^2 \left[ \frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{\text{rad},0}}{x^4} + \Omega_{\text{vac},0} + \frac{\Omega_{f,0}}{x^8} \right],$$

which can be rewritten as

$$x \frac{dx}{dt} = H_0 \sqrt{\Omega_{m,0}x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0}x^4 + \frac{\Omega_{f,0}}{x^4}}.$$

From here the derivation is identical to that in part (c), leading to

$$t_d = \frac{1}{H_0} \int_0^{x_d} \frac{x \, dx}{\sqrt{\Omega_{m,0}x + \Omega_{\text{rad},0} + \Omega_{\text{vac},0}x^4 + \frac{\Omega_{f,0}}{x^4}}} ,$$

which can also be written more neatly as

$$t_d = \frac{1}{H_0} \int_0^{x_d} \frac{x^3 \, dx}{\sqrt{\Omega_{m,0}x^5 + \Omega_{\text{rad},0}x^4 + \Omega_{\text{vac},0}x^8 + \Omega_{f,0}}}.$$