NON-EUCLIDEAN SPACES
CLOSED AND OPEN UNIVERSES
Announcements

★ Problem Set 5 is due Friday, 4:00 pm.
Summary of Lecture 10: Surface of a Sphere

\[ x^2 + y^2 + z^2 = R^2. \]
Polar Coordinates:

\[ x = R \sin \theta \cos \phi \]
\[ y = R \sin \theta \sin \phi \]
\[ z = R \cos \theta \]
Varying $\theta$:

$$ds = R\,d\theta$$
Varying $\phi$: 

$$ ds = R \sin \theta \, d\phi $$
Varying $\theta$ and $\phi$

Varying $\theta$: $ds = R \, d\theta$

Varying $\phi$: $ds = R \sin \theta \, d\phi$

$ds^2 = R^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right)$
A Closed Three-Dimensional Space

\[ x^2 + y^2 + z^2 + w^2 = R^2 \]

\[
\begin{align*}
  x &= R \sin \psi \sin \theta \cos \phi \\
  y &= R \sin \psi \sin \theta \sin \phi \\
  z &= R \sin \psi \cos \theta \\
  w &= R \cos \psi ,
\end{align*}
\]

\[ ds = R \, d\psi \]
Metric for the Closed 3D Space

Varying $\psi$: \[ ds = R \, d\psi \]

Varying $\theta$ or $\phi$: \[ ds^2 = R^2 \sin^2 \psi (d\theta^2 + \sin^2 \theta \, d\phi^2) \]

If the variations are orthogonal to each other, then

\[ ds^2 = R^2 \left[ d\psi^2 + \sin^2 \psi \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right] \]
Proof of Orthogonality of Variations

Let $\vec{dr}_\psi = \text{displacement of point when } \psi \text{ is changed to } \psi + d\psi$.
Let $\vec{dr}_\theta = \text{displacement of point when } \theta \text{ is changed to } \theta + d\theta$.

★ $\vec{dr}_\theta \text{ has no } w\text{-component} \implies \vec{dr}_\psi \cdot \vec{dr}_\theta = \vec{dr}_\psi^{(3)} \cdot \vec{dr}_\theta^{(3)}$, where (3) denotes the projection into the $x$-$y$-$z$ subspace.

★ $\vec{dr}_\psi^{(3)}$ is radial; $\vec{dr}_\theta^{(3)}$ is tangential

\[ \implies \vec{dr}_\psi^{(3)} \cdot \vec{dr}_\theta^{(3)} = 0 \]
Implications of General Relativity

\[ ds^2 = R^2 \left[ d\psi^2 + \sin^2 \psi \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right], \text{ where } R \text{ is radius of curvature.} \]

\[ \text{According to GR, matter causes space to curve.} \]

\[ R \text{ cannot be arbitrary. Instead, } R^2(t) = \frac{a^2(t)}{k}. \]

\[ \text{Finally,} \]

\[ ds^2 = a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right\}, \]

where \( r = \frac{\sin \psi}{\sqrt{k}} \). Called the Robertson-Walker metric.