8.286 Lecture 9
October 10, 2018

DYNAMICS OF HOMOGENEOUS EXPANSION, PART IV
Summary of Lecture 8

Age of a Flat Matter-Dominated Universe:

\[ a(t) \propto t^{2/3} \quad \implies \quad t = \frac{2}{3} H^{-1} \]

For \( H = 67.7 \pm 0.5 \text{ km-s}^{-1}\text{-Mpc}^{-1} \), age = 9.56 – 9.70 billion years — but stars are older. Conclusion: our universe is nearly flat, but not matter-dominated.
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The Big Bang Singularity: \( a(0) = 0 \), with infinite density, is a feature of our model, but not necessarily the real universe.
Horizon Distance: the present distance of the furthest particles from which light has had time to reach us.

\[ \ell_{\text{phys,horizon}}(t) = a(t) \int_0^t \frac{c}{a(t')} \, dt'. \]

\[ a(t) \propto t^{2/3} \quad \implies \quad \ell_{\text{phys,horizon}} = 3ct = 2cH^{-1}. \]
Equations for a Matter-Dominated Universe

(“Matter-dominated” = dominated by nonrelativistic matter.)

Friedmann equations:

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3} G \rho - \frac{kc^2}{a^2},
\]

\[
\ddot{a} = -\frac{4\pi}{3} G \rho(t) a.
\]

Matter conservation:

\[
\rho(t) \propto \frac{1}{a^3(t)}, \text{ or } \rho(t) = \left[ \frac{a(t_1)}{a(t)} \right]^3 \rho(t_1) \text{ for any } t_1.
\]

Any two of the above equations can allow us to find the third.
Evolution of a Closed Universe

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3} G \rho - \frac{k c^2}{a^2}, \quad \rho(t)a^3(t) = \text{constant}, \quad k > 0.
\]

Recall \([a(t)] = \text{meter/notch}, [k] = 1/\text{notch}^2\).

Define new variables:

\[
\tilde{a}(t) \equiv \frac{a(t)}{\sqrt{k}} , \quad \tilde{t} \equiv ct \quad \text{(both with units of distance)}
\]

Multiplying Friedmann eq by \(a^2/(kc^2)\):

\[
\frac{1}{kc^2} \left( \frac{da}{dt} \right)^2 = \frac{8\pi}{3} \frac{G \rho a^2}{kc^2} - 1.
\]
\[ \frac{1}{k c^2} \left( \frac{da}{dt} \right)^2 = \frac{8\pi}{3} \frac{G \rho a^2}{k c^2} - 1 \]

\[ = \frac{8\pi}{3} \frac{G \rho a^3}{k^{3/2} c^2} \frac{\sqrt{k}}{a} - 1 . \]

(4.15)

Rewrite as

\[ \left( \frac{d\tilde{a}}{dt} \right)^2 = \frac{2\alpha}{\tilde{a}} - 1 , \]

where

\[ \alpha \equiv \frac{4\pi}{3} \frac{G \rho \tilde{a}^3}{c^2} . \]

[\alpha] = meter. \( \alpha \) is constant, since \( \rho a^2 \) is constant.
\[
\left( \frac{d\tilde{a}}{d\tilde{t}} \right)^2 = \frac{2\alpha}{\tilde{a}} - 1 \quad \Longrightarrow \quad d\tilde{t} = \frac{\tilde{a} \, d\tilde{a}}{\sqrt{2\alpha\tilde{a} - \tilde{a}^2}}.
\]

Then

\[
\tilde{t}_f = \int_{0}^{\tilde{t}_f} d\tilde{t} = \int_{0}^{\tilde{a}_f} \frac{\tilde{a} \, d\tilde{a}}{\sqrt{2\alpha\tilde{a} - \tilde{a}^2}},
\]

where \( \tilde{t}_f \) is an arbitrary choice for a “final time” for the calculation, and \( \tilde{a}_f \) is the value of \( \tilde{a} \) at time \( \tilde{t}_f \).
Evolution of a Closed Universe

\[ ct = \alpha (\theta - \sin \theta) , \]

\[ \frac{a}{\sqrt{k}} = \alpha (1 - \cos \theta) . \]
Evolution of a Closed Universe

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Evolution of a Closed Universe

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\[ \frac{a}{\sqrt{k}} = \alpha(1 - \cos \theta) . \]

\[ t = \frac{\Omega}{2|H|(\Omega - 1)^{3/2}} \left\{ \arcsin \left( \pm \frac{2\sqrt{\Omega - 1}}{\Omega} \right) \mp \frac{2\sqrt{\Omega - 1}}{\Omega} \right\} . \]
\[ t = \frac{\Omega}{2|H|(\Omega - 1)^{3/2}} \left\{ \arcsin \left( \pm \frac{2\sqrt{\Omega - 1}}{\Omega} \right) \mp \frac{2\sqrt{\Omega - 1}}{\Omega} \right\} . \]

<table>
<thead>
<tr>
<th>Quadrant</th>
<th>Phase</th>
<th>( \Omega )</th>
<th>Sign Choice</th>
<th>( \sin^{-1}(\cdot) )</th>
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<tbody>
<tr>
<td>1</td>
<td>Expanding</td>
<td>1 to 2</td>
<td>Upper</td>
<td>0 to ( \frac{\pi}{2} )</td>
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<tr>
<td>2</td>
<td>Expanding</td>
<td>2 to ( \infty )</td>
<td>Upper</td>
<td>( \frac{\pi}{2} ) to ( \pi )</td>
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<td>( \infty ) to 2</td>
<td>Lower</td>
<td>( \pi ) to ( \frac{3\pi}{2} )</td>
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<tr>
<td>4</td>
<td>Contracting</td>
<td>2 to 1</td>
<td>Lower</td>
<td>( \frac{3\pi}{2} ) to ( 2\pi )</td>
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