1) **f-I curve for the leaky (LIF) integrate-and-fire neuron.** As we discussed in class, a highly simplified description of neuron dynamics is given by the Leaky Integrate-and-Fire (LIF) model. The subthreshold neural voltage is given by:

\[
C_m \frac{dV(t)}{dt} = -g_m(V - V_m) + I_{inj}
\]

where \(C_m\) is the membrane capacitance, \(g_m\) a fixed (leak) conductance that maintains the cell at a resting potential \(V_m\) in the absence of inputs, and \(I_{inj}\) is an injected current. This equation is augmented by a threshold condition: when \(V\) reaches a threshold \(V_\theta\), the voltage is immediately reset to \(V_{reset} < V_\theta\). At that point, the neuron is considered to have spiked.

a. Derive analytically the critical current \(I_{inj}^*\) below which the neuron will never fire and above which it will as a function of the parameters \(g_m, V_m\). Explain in words the dependence of \(I_{inj}^*\) on \(g_m\): why does it vary the way it does?

b. Derive the the f-I curve of the neuron: that is, derive the relationship between \(I_{inj}\) and the firing rate \(\nu\) of the LIF neuron. (Hint: Solve for \(V(t)\) at a constant \(I_{inj}\) and solve for how long it takes to go from \(V = V_{reset}\) to \(V_\theta\). Plot \(\nu\) versus \(I_{inj}\), using parameters \(V_m = -60\) mV, \(g_m = 0.1\) mS/cm\(^2\), \(C_m = 1\) µF/cm\(^2\), \(V_E = 0\) mV, \(V_\theta = -50\) mV, \(V_{reset} = -55\) mV.

c. Use the expansion \(\log(1 + \epsilon) \approx \epsilon\) (which is valid when \(\epsilon\) is small) on your expressions from b., to show that the f-I curve is approximately linear. Add this approximate linear expression for the f-I curve to your plot in b. to show that the linear approximation is good for large values of \(I_{inj}\).

2) **Numerical simulation of the LIF neuron and assessment of the method-of-averaging to obtain rate-based expressions.** Consider Equation 1 (and the reset condition) together with the following equation for synaptic activation:

\[
\tau \frac{ds}{dt} = -s + \sum_a \delta(t - t_a),
\]

where the sum over \(a\) in the synaptic equation is over all the times \(t_a\) at which the neuron spikes.

a. Numerically integrate Equations (1)-(2) to obtain the time-evolution of the voltage and synaptic activation of a LIF neuron, over a 1 second interval. (To
numerically integrate: replace the derivative by a finite-time difference over an interval $\Delta t$ according to the “Euler method”, as we saw in class, and iterate. Use $\Delta t = 0.1$ ms.) Use parameters from Problem 1. Initial conditions: $s(0)=0$ and $V(0) = V_{\text{reset}}$. At $t = 1$ s, step $I_{\text{inj}}$ from 0 to a value that produces spiking at approximately 60 Hz; hold this step until $t = 2$, then return $I_{\text{inj}}$ to 0. Plot the simulated $V(t)$ and $s(t)$ versus time in ms, with $s(t)$ simulated under three conditions: $\tau_s = 10$ ms, $\tau_s = 50$ ms, and $\tau_s = 100$ ms.

b. As in class, averaging over the fast spiking variable – specifically over one interspike interval – reduces this pair of equations to a single differential equation for $s$ in terms of the firing rate $\nu(I_{\text{inj}})$ of the neuron. Assume that $\tau_s$ is large compared to the interval between spikes in the neuron. Show the rate-based equation results for $s(t)$ on top of $s(t)$ obtained from the spiking simulations in a. above. When is the rate-based equation more accurate?

3) **Single neuron with two time scales.** Consider a single neuron with two selfsynapses, one excitatory and one inhibitory, with different characteristic time scales. The rate of the neuron is given by:

$$r = \alpha(s_1 - s_2), \quad \alpha > 0$$

and the synaptic activities:

$$\tau_1 \dot{s}_1 = -s_1 + r$$
$$\tau_2 \dot{s}_2 = -s_2 + r$$

Here $\tau_1 > 0$, $\tau_2 > 0$ are the characteristic time scales.

a. Write the system dynamics in a two dimensional matrix form, $\dot{s} = As$. Calculate the eigenvalues of the matrix $A$ as a function of $\alpha, \tau_1, \tau_2$. What are the possible fixed points of the system?

b. For $\alpha = 1$, what will be the nature of the dynamics (exponential decay/growth, oscillations...) in case of $\tau_1 > \tau_2$? and for $\tau_2 > \tau_1$? Give an intuitive explanation for the difference between these cases. Draw qualitatively (by hand or using a computer) a representing solution trajectory in the $s_1s_2$ plane for each of the cases.

c. For $\alpha \gg 1$, what is the condition for the system to be stable? Explain.

d. Now, assume that the synaptic activities have different amplitudes in the rate equation:

$$r = \alpha_1 s_1 - \alpha_2 s_2, \quad \alpha_1, \alpha_2 > 0$$
What is the condition on \( \alpha_1, \alpha_2 \) in order for the system to have a continuum of fixed points? Write an equation (\( s_1 \) as a function of \( s_2 \)) describing the continuum of fixed points when the condition you found is realized, and draw it qualitatively.

e. Under the condition found in the last sub question, find a condition for the continuum of fixed points to be attractive.

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