CHAPTER 4 Stress, Strain, Stress-Strain

4.1 The Concept of Stress — An Introduction

We have talked about internal forces, distributed them uniformly over an area and they became a *normal stress* acting *perpendicular* to some internal surface, or a *shear stress* acting *tangentially, in plane*. Up to now, we have said little about how these normal and shear stresses might vary with position throughout a solid.[1] Up to now, the choice of planes, their orientation within a solid, was dictated by the geometry of the solid and the nature of the loading. We have said nothing about how these stress components might change if we looked at a set of planes of another orientation.

Now we consider a more general situation, an arbitrarily shaped solid. We are going to lift our gaze up from the world of crude structural elements such as truss bars in tension, shafts in torsion, or beams in bending to view these "solids" from a more abstract perspective. They all become special cases of more general stuff we call a solid continuum.

We will address two questions:

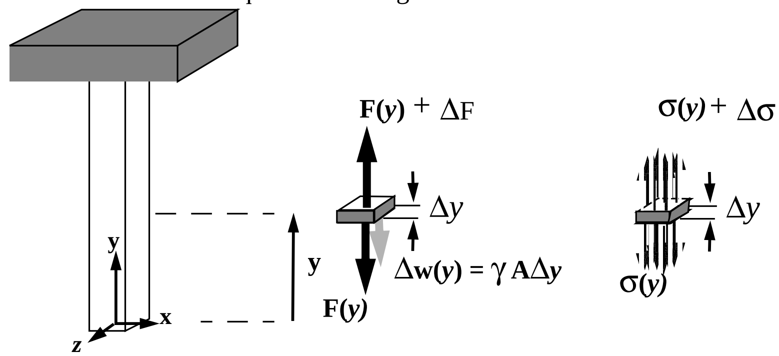
- How might stresses vary from one point to another throughout a continuum;
- How do the normal and shear components of stress acting on a plane at a given point change as we change the orientation of the plane at the point.

The first bullet introduces the notion of *stress field;* the second concerns the *transformation of components of stress at a point*.

To begin with the first bullet we re-examine the case of a bar suspended vertically and loaded by its own weight, a case considered in section 3.2, page 62. (Note, I have changed the orientation of the reference

1. The beam is the one exception. There we explored how different normal stress distributions over a rectangular cross-section could be equivalent to a bending moment and zero resultant force.

axes). We will construct a differential equation which governs how the axial stress varies as we move up and



down the bar. We will solve this differential equation, not forgetting to apply an appropriate *boundary condition* and determine the axial stress field.

We see that for equilibrium of the differential element of the bar, of planar cross-sectional area A and of weight density γ, we have

$$F + \Delta F - \gamma \cdot A \cdot \Delta y - F = 0$$

If we assume the tensile force is uniformly distributed over the cross-sectional area, and dividing by the area (which does not change with the independent spatial coordinate y) we can write

$$\sigma + \Delta\sigma - \gamma \cdot \Delta y - \sigma = 0 \qquad \text{where} \qquad \sigma \equiv F/A$$

Chanting "...going to the limit, letting Δy go to zero", we obtain a differential equation fixing how σ(y), a function of y, varies throughout our continuum, namely

$$\frac{d\sigma}{dy} - \gamma = 0$$

We solve this ordinary differential equation easily, integrating once and obtain

$$\sigma(y) = \gamma \cdot y + \text{Constant}$$

The *Constant* is fixed by a prescribed condition at some *y* surface; If the end of the bar is *stress free*, we indicate this writing

$$\text{at } y = 0 \qquad \sigma = 0$$

so

$$\sigma(y) = \gamma \cdot y$$

If, on another occasion, a weight of magnitude P_0 is suspended from the free end, we would have

$$\text{at } y = 0 \qquad \sigma = P_0/A$$

and

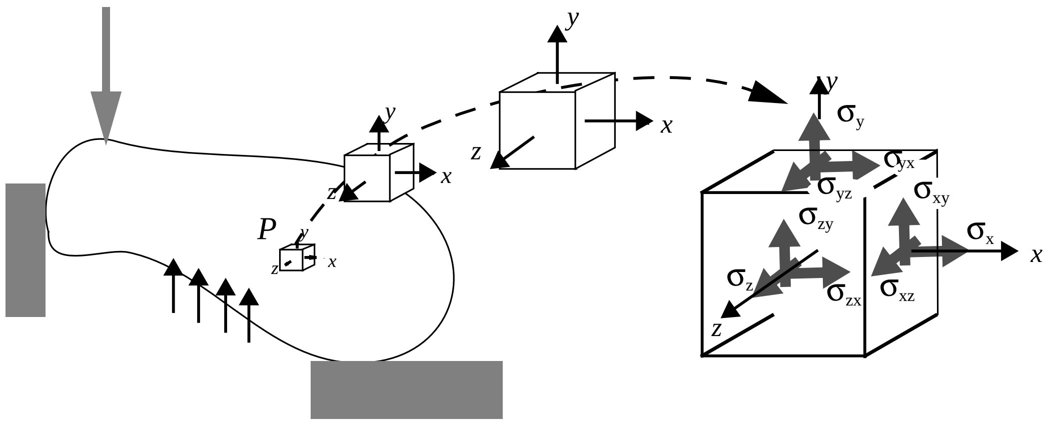
$$\sigma(y) = \gamma \cdot y + P_0/A$$

Here then are two stress fields for two different loading conditions[1]. Each stress field describes how the normal stress $\sigma(x,y,z)$ varies throughout the continuum at every point in the continuum. I show the stress as a function of *x* and *z* as well as y to emphasize that we can evaluate its value at *every point* in the continuum, although it only varies with *y*. That the stress does not vary with *x* and *z* was implied when we

1. A third loading condition is obtained by setting the weight density γ to zero; our bar then is assumed weightless relative to the end-load P_0.

stipulated or assumed that the internal force, *F*, acting upon any *y* plane was uniformly distributed over that plane. This example is a special case in another way; not only is it one-dimensional in its dependence upon spatial position, but it is the simplest example of stress at a point in that it is described fully by a single component of stress, the normal stress acting on a plane perpendicular to the y axis.

The figure below is meant to illustrate a more general, indeed, the most general state of stress at a point. It requires some explanation:



The odd looking structural element, fixed to the ground at bottom and to the left, and carrying what appears to be a uniformly distributed load over a portion of its bottom and a concentrated load on its top, is meant to symbolize an arbitrarily loaded, arbitrarily constrained, arbitrarily shaped solid continuum. It could be a beam, a truss, a thin-walled cylinder though it looks more like a potato — which too is a solid continuum. At any arbitrarily chosen point inside this object we can ask about the internal stress state. But *what stress component*?

For there are more than one; in fact they come in sets of three. One set acts upon what we call an *x plane*, another upon a *y plane*, a third set upon a *z plane.* Which plane is which is defined by its *normal*: An *x plane has its normal in the x direction*, etc. Each set includes three scalar components, one *normal stress component* acting perpendicular to its reference plane, with its direction along one coordinate axis, and two *shear stress components* acting *in plane* in the direction of the other two coordinate axes.

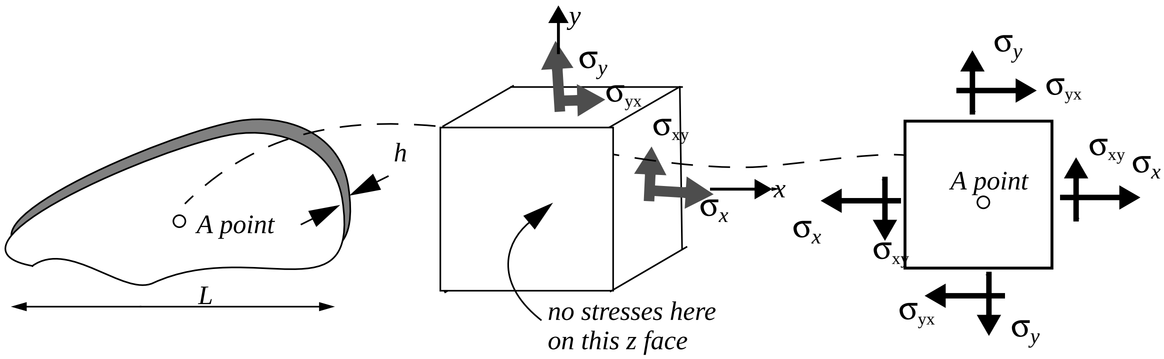
That's a grand total of *nine stress components to define the stress at a point.* To fully define the stress field throughout a continuum you need to specify how these nine scalar components vary from one point in the continuum to another. That's a tall order.

Fortunately, equilibrium requirements applied to a differential element of the continuum, what we will call a "micro-equilibrium" consideration, will reduce the number of independent stress components at a point from nine to six. We will find that the shear stress component σ_{xy} acting on the *x* face must equal its neighbor around the corner σ_{yx} acting on the *y* face and that $\sigma_{zy} = \sigma_{yz}$ and $\sigma_{xz} = \sigma_{zx}$ accordingly.

Fortunately too, in most of the engineering structures you will encounter, diagnose or design, only two or three of these now six components will matter, will be *significant*. And, as in the example just treated of a bar suspended vertically and loaded by its own weight and/or by an end load, often variations of the stress components in one, or more, of the three coordinate directions may be uniform. But perhaps the most important simplification is a simplification in modeling, made at the outset of our encounter. One particularly useful model, applicable to many structural elements is called Plane Stress and, as you might infer from the label alone, it restricts our attention to variations of stress in two dimensions.

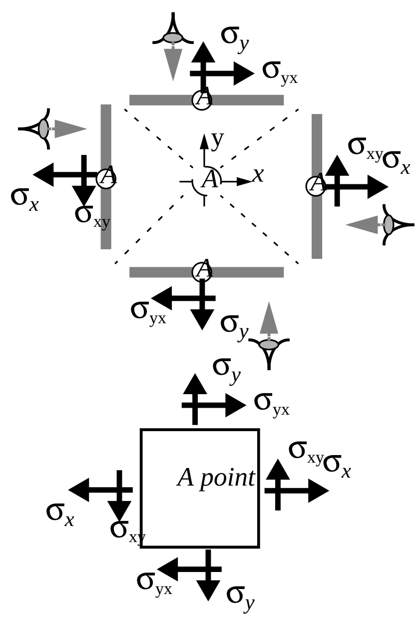
4.2 Plane Stress

If we assume our continuum has the form of a thin plate of uniform thickness but of arbitrary closed contour in the *x-y* plane, our previous arbitrarily loaded, arbitrarily constrained continuum (we don't show these again) takes the planar form below.

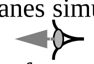


Because the plate is thin in the z direction, ($h/L << 1$) we will assume that variations of the stress components with z is uniform or, in other words, our stress components will be at most functions of *x* and *y*, $\sigma(x,y)$. We also take it that the *z boundary planes are unloaded, stressfree*. These two together imply that the set of three "z" stress components that act upon *any* arbitrarily located z plane within the interior must also vanish. We will also take advantage of the micro-equilibrium consequences, yet to be explored but noted previously, and set σ_{yz} and σ_{xz} to zero. Our state of stress at a point is then as it is shown on the exploded view of the point - the block in the middle of the figure - and again from the point of view of looking normal to a z plane at the far right. This special model is called **Plane Stress**.

A Word about Sign Convention:



The figure at the far right seems to include more stress components than necessary; after all, if, in modeling, we eliminate the stress components acting on a *z* face and σ_{yz} and σ_{xz} as well, that should leave, at most, four components acting on the *x* and *y* faces. Yet there appear to be eight in the figure. No, there are only at most four components; we must learn to read the figure.

To do so, we make use of another sketch of *stress at a point*, the point *A*. The figure at the top is meant to indicated that we are looking at four faces or planes simultaneously. When we look at the *x face* from the right  we are looking at the stress components on *a positive x face* — it has its *outward normal in the positive x direction* — and a *positive normal stress*, by convention, is *directed in the positive x direction*. A *positive shear stress component*, acting in plane, also acts, by convention, *in a positive coordinate direction* - in this case the positive y direction.

On the positive y face, we follow the same convention; *a positive σ_y acts on a positive y face in the positive y coordinate direction; a positive σ_{yx} acts on a positive y face in the positive x coordinate direction.*

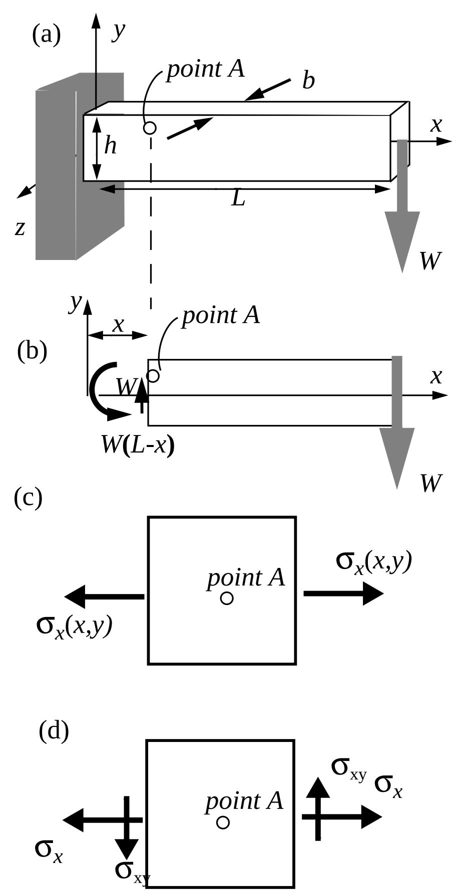
We emphasize that we are looking at a point, point A, in these figures. More precisely we are looking at two mutually perpendicular planes intersecting at the point and from two vantage points in each case. We draw these two views of the two planes as four planes in order to more clearly illustrate our sign convention. But you ought to imagine the square having zero height and width: the σ_x acting to the *left*, in the *negative x direction*, upon the *negative x face* at the left, with its *outward normal pointing in the negative x direction* is *a positive component* at the point, the **equal and opposite reaction to the σ_x** acting to the right, in the *positive x direction*, upon the *positive x face* at the right, with its *outward normal pointing in the positive x direction*. Both are positive as shown; **both are the same quantity**. So too the shear stress component σ_{xy} shown acting down, *in the negative y direction*, on the *negative x face* is the equal and opposite internal reaction to σ_{xy} shown acting up, *in the positive y direction*, on the *positive x face*

A general statement of our sign convention, which holds for all nine components of stress, even in 3D, is as follows: **A positive component of stress acts on a positive face in a positive coordinate direction** *or* **on a negative face in a negative coordinate direction.**

***

An Example:

We might model the end-loaded cantilever with relatively thin rectangular cross-section as a plane stress problem. In this, b is the “thin” dimension, i.e., $b/L < 1$.



If we assume a normal stress distribution *over an x face* is proportional to some odd power of y, as we did in section 3.2, exercise 3.5, our state of stress at a point might look like that shown in figure (c). In this, σ_x would have the form

$$\sigma_x(x, y) = C(n, b, h) \cdot W(L - x)y^n$$

where C(n,b,h) is a constant which depends upon the cross-sectional dimensions of the beam and the odd exponent n. (See page 57). The factor W(L-x) is the magnitude of the internal bending moment at the location x measured from the root. See figure (b).

But this is only one component of our stress field. What are the other eight components of stress at point A?

Our plane stress model allows us to claim that the three *z face* components are zero and if we take σ_{yz} and σ_{xz} to be zero, that still leaves σ_{xy}, σ_{yx}, and σ_y in addition to σ_x.

Now, in fact, we are doomed from the start; we know that the problem is statically indeterminate so we are not going to be able to construct a unique solution to the equilibrium requirements and specify all nine components of our stress field. Still we can experiment taking advantage of the indeterminacy at our disposal.

For example, we know that a shear force of magnitude W acts at any x section. In this case it does not vary with x. We might assume, then, that the shear force is uniformly distributed over the cross-section and set

$\sigma_{xy} = -W/(bh)$ Our stress at a point at point A would then look like figure (d).

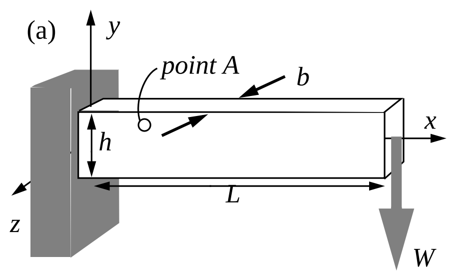
We could, of course, posit other shear stress distributions at any x station, e.g., some function like $\sigma_{xy} = \text{Constant} \cdot y^m$ where m is an integer and the constant is determined from the requirement that the resultant force due to this shear stress distribution over the cross section must be W.

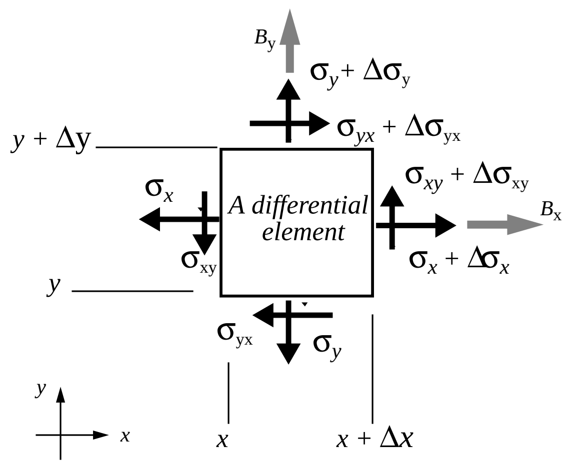
But this is about as far as we can go with our fabrication of a stress field. There are other matters, matters of equilibrium at the micro scale, that must be addressed prerequisite to establishing a useful description of the stress field within a continuum. Yet even with this the problem remains statically indeterminate. We have yet to consider the requirements of continuity of displacement and compatibility of deformation.

4.3 Stress Fields & “Micro” Equilibrium

We return and pick up on the example of a bar hanging under its own weight and explore the consequences of equilibrium when applied to a differential element of an arbitrarily shaped two dimensional, plane stress continuum. We call this “micro” equilibrium considerations to distinguish it from the “macro” equilibrium considerations of the last chapter. There we isolated large chunks of structure.

Think now, of a differential element in 2D; for example, of the cantilever beam: We show such on the right. Note now we are no longer focused on two intersecting, perpendicular planes at a point but on a differential element of the continuum. Now we see that the stress components may very well be different on the two *x* faces and on the two *y* faces.





We allow the x face components, and those on the two y faces to change as we move from *x* to $x+\Delta x$ (holding y constant) and from *y* to $y+\Delta \mathbf{y}$ (holding *x* constant).

We show two other arrows on the figure, B_x and B_y. These are meant to represent the x and y components of what is called a *body force.* A body force is any externally applied force acting on each element of volume of the continuum. It is thus a force per unit volume. For example, if we need consider the weight of the beam, B_y would be just

$$_y = -\gamma \qquad \text{where} \qquad \gamma = \text{the weight density}$$

where the negative sign is necessary because we take a positive component of the body force vector to be in a positive coordinate direction.

B_x would be taken as zero.

We now consider force and moment equilibrium for this differential element, our micro isolation. We sum forces in the *x* direction which will include the shear stress component σ_{yx}, acting on the ***y*** face in the *x* direction as well as the normal stress component σ_x acting on the *x* faces. But note that these components are not forces; to figure their contribution to the equilibrium requirement, we must factor in the areas upon which they act.

I present just the results of the limiting process which, we note, since all components may be functions of both x and y, brings partial derivatives into the picture.

$$\sum \text{Forces in the x direction} \rightarrow \qquad \frac{\partial\sigma_x}{\partial x} + \frac{\partial\sigma_{yx}}{\partial y} + B_x = 0$$

$$\sum \text{Forces in the y direction} \rightarrow \qquad \frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + B_y = 0$$

$$\sum \text{Moments about the center of the element} \rightarrow \qquad \sigma_{yx} = \sigma_{xy}$$

For example, the change in the stress component $\Delta\sigma_x$ may be written

$$\Delta\sigma_x = \frac{\partial\sigma_x}{\partial x} \cdot \Delta x$$

and the force due to this "unbalanced" component in the x direction is

$$\frac{\partial\sigma_x}{\partial x} \cdot \Delta x \cdot (\Delta y \Delta z)$$

where the product, $\Delta y\ \Delta z$, is just the differential area of the *x face*.

The contribution of the body force (per unit volume) to the sum of force components in the x direction will be $B_x(\Delta x\Delta y\Delta z)$ where the product of deltas is just the differential volume of the element. We see that this product will be a common factor in all terms entering into the equations of force equilibrium in the x and y directions.

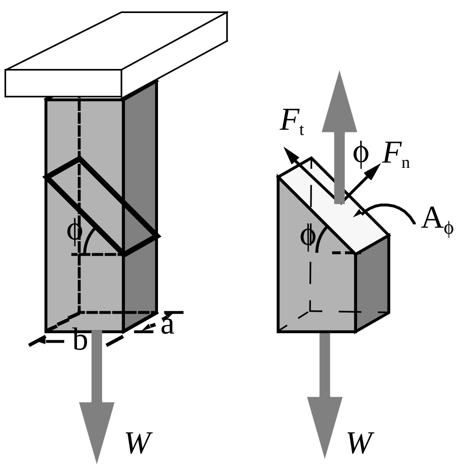
The last equation of moment equilibrium shows that, as we forecast, the shear stress component on the *y f*ace must equal the shear stress component acting on the *x* face. The differential changes in the shear stress components are of lower order and drop out of consideration in the limiting process, as we take Δx and Δy to zero.

We leave this topic to the side for the moment and turn now to the second item on our agenda — the transformation of components of stress at a point.

4.4 Transformation of Components of Stress

While realizing that stress can vary from point to point in a solid continuum, and even if we can find the solution to the micro equations of equilibrium for the normal and shear stress components throughout, that still may not be enough to define when the solid might fracture or yield when subject to some applied and prescribed load. The reason is that the values of the stress components at a point change with respect to the orientation of the axis, or what is the same, the orientation of the planes we choose to view at the point.

To proceed, we consider our simple example of a bar suspended vertically but now take the weight of the bar to be negligible relative to the weight suspended at its free end and explore how the normal and shear stress components at a point vary as we change, not the position of the point, but the orientation of a plane through the point.



Exercise 4.1 –The solid column of rectangular cross section measuring $a \times b$ supports a weight *W. Show that* both a normal stress and a shear stress must act on any inclined interior face. Determine their respective values assuming that both are uniformly distributed over the area of the inclined face. Express your estimates in terms of the ratio *(W/ab*) and the angle ϕ.

For equilibrium of the isolation of a section of the column shown at the right, a force equal to the suspended weight (we neglect the weight of the column itself) must act upward. We show an equivalent force system — or, if you like, its components consisting of two perpendicular forces, one directed normal to the inclined plane, the other with its line of action in the plane inclined at the angle ϕ. We have

$$F_n = W \cdot \cos\phi \qquad \text{and} \qquad F_t = W \cdot \sin\phi$$

Now if we assume these are distributed uniformly over the section, we can construct an estimate of the normal stress and the shear stress acting on the inclined face. But first we must establish the area of the inclined face $A\phi$. From the geometry of the figure we see that the length of the inclined plane is $b/\cos\phi$ so the area is $A_\varphi = (ab)/(\cos\phi)$

With this we write the normal and shear stress components as

$$\sigma_n = F_n / A_\phi = \left(\frac{W}{ab}\right) \cdot \cos\phi^2$$

and

$$\sigma_t = F_t / A_\phi = \left(\frac{W}{ab}\right) \cdot \cos\phi \sin\phi$$

These results clearly illustrate how the values for the normal and shear stress components of a force distributed over a plane inside of an object depends upon how you look at the point inside the object in the sense that **the values of the shear and normal stresses at a point within a continuum depend upon the orientation of the plane you have chosen to view.**

Why would anyone want to look at some arbitrarily oriented plane in an object, seeking the normal and shear stresses acting on the plane? Why do we ask you to learn how to figure out what the stress components on such a plane might be?

The answer goes as follows: One of our main concerns as a designer of structures is failure —fracture or excessive deformation of what we propose be built and fabricated. Now many kinds of failures initiate at a local, microscopic level. A minute imperfection at a point in a beam where the local stress is very high initiates fracture or plastic deformation, for example. Our quest then is to figure out where, at what points in a structural element, the normal and shear stress components achieve their maximum values. But we have just seen how these values depend upon the way we look at a point, that is, upon the orientation of the plane we choose to inspect. To ensure we have found the maximum normal stress at a point for example, we would then have to inspect every possible orientation of a plane passing through the point.[1]

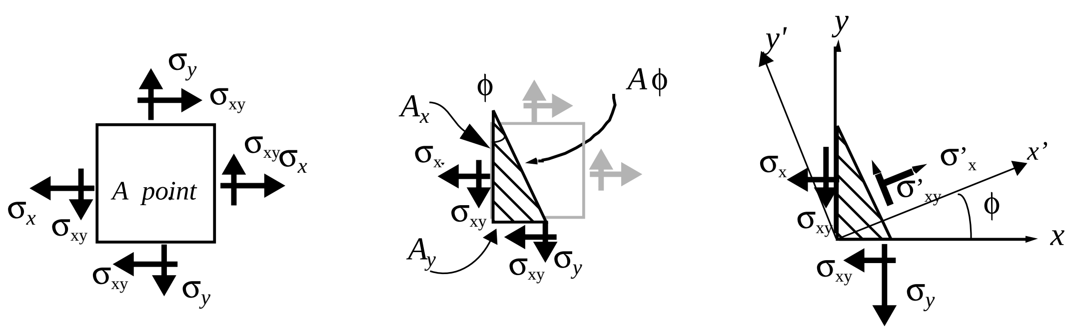
This seems a formidable task. But before taking it on, we pose a prior question:

Exercise 4.2 What do you *need to know* in order to determine the normal and shear stress components acting upon an *arbitrarily oriented plane at a point* in a fully three dimensional object?

The answer is what we might anticipate from our original definition of six stress components for if we know these six scalar quantities[2], the three normal stress components σ_x, σ_y, and σ_z, and the three shear stress components σ_{xy}, σ_{yz}, and σ_{xz}, then we can find the normal and shear stress components acting upon an arbitrarily oriented plane at the point. That is the answer to our need to know question.

To show this, we derive a set of equations that will enable you to do this. But note: we take the six stress components relative to the three orthogonal, let's call them, *x,y,z* planes as given, as known quantities. Furthermore, again we restrict our attention to two dimensions - the case of Plane Stress. That is we say that the components of stress acting on one of the planes at the point - we take the *z* planes - are zero. This is a good approximation for certain objects — those which are *thin* in the *z* direction relative to structural element's dimensions in the *x-y* plane. It also, makes our derivation a bit less tedious, though there is nothing conceptual complex about carrying it through for three dimensions, once we have it for two.

In two dimensions we can draw a simpler picture of *the state of stress at a point*. We are not talking differential element here but of stress at a point. The figure below shows an arbitrarily oriented plane, defined by its normal, the *x'* axis, inclined at an angle ϕ to the horizontal. In this two dimensional state of



stress we have but three scalar components to specify to fully define the state of stress at a point: σ_x, σ_y and $\sigma_{yx} = \sigma_{xy}$. Knowing these three numbers, we can determine the normal and shear stress components acting on any plane defined by the orientation ϕ as follows.

1. Much as we have done in the preceding exercise. Equations 68 and 69 show that the maximum normal stress acts on the horizontal plane, defined by ϕ =0. The maximum shear stress, on the other hand acts on a plane oriented at 45^o to the horizontal. The factor cosϕ sin ϕ has a maximum at $\phi = 45^o$.

2. We take advantage of moment equilibrium and take $\sigma_{yx} = \sigma_{xy}$, $\sigma_{zx} = \sigma_{xz}$, and $\sigma_{zy} = \sigma_{yz}$.

Consider equilibrium of the shaded wedge shown. Here we let $A\phi$ designate the area of the inclined face at a point, A_x and A_y the areas of the x face with its outward normal pointing in the $-x$ direction and of the y face with its outward normal pointing in the $-y$ direction respectively. In this we take a unit depth into the paper. We have

$$A_x = A_\phi \cdot \cos\phi \qquad \text{and} \qquad A_y = A_\phi \cdot \sin\phi$$

That takes care of the relative areas. Now for *force* equilibrium, in the x and y directions we must have:

$$-\sigma_x \cdot A_x - \sigma_{xy} \cdot A_y + (\sigma'_x \cdot \cos\phi - \sigma'_{xy} \cdot \sin\phi) \cdot A_\phi = 0$$

and

-

$$-\sigma_{xy} \cdot A_x - \sigma_y \cdot A_y + (\sigma'_x \cdot \sin\phi + \sigma'_{xy} \cdot \cos\phi) \cdot A_\phi = 0$$

If we multiply the first by cosϕ, the second by sinϕ and add the two we can eliminate σ'_{xy}. We obtain

$$\sigma'_x A_\phi - \sigma_x \cos\phi A_x - \sigma_{xy} \cos\phi A_y - \sigma_{xy} \sin\phi A_x - \sigma_y \sin\phi A_y = 0$$

which, upon expressing the areas of the x,y faces in terms of the area of the inclined face, can be written (noting $A\phi$ becomes a common factor).

$$\sigma'_x = \sigma_x \cos\phi^2 + \sigma_y \sin\phi^2 + 2\sigma_{xy} \sin\phi \cos\phi$$

In much the same way, multiplying the first equilibrium equation by sinϕ, the second by cosϕ but subtracting rather than adding you will obtain eventually

$$\sigma'_{xy} = (\sigma_y - \sigma_x) \sin\phi \cos\phi + \sigma_{xy}(\cos\phi^2 - \sin\phi^2)$$

We deduce the normal stress component acting on the y' face of this rotated frame by replacing ϕ in our equation for σ'_x by $\phi + \pi/2$. We obtain in this way:

$$\sigma'_y = \sigma_y \cos\phi^2 + \sigma_x \sin\phi^2 - 2\sigma_{xy} \sin\phi \cos\phi$$

The three transformation equations for the three components of stress at a point can be expressed, using the double angle formula for the cosine and the sine, as

$$\sigma'_x = \left[\frac{(\sigma_x + \sigma_y)}{2}\right] + \left[\frac{(\sigma_x - \sigma_y)}{2}\right] \cdot \cos 2\phi + \sigma_{xy} \sin 2\phi$$

$$\sigma'_y = \left[\frac{(\sigma_x + \sigma_y)}{2}\right] - \left[\frac{(\sigma_x - \sigma_y)}{2}\right] \cdot \cos 2\phi - \sigma_{xy} \sin 2\phi$$

$$\sigma'_{xy} = -\left[\frac{(\sigma_x - \sigma_y)}{2}\right] \cdot \sin 2\phi + \sigma_{xy} \cos 2\phi$$

Here we have the equations to do what we said we could do. Think of the set as a machine: You input the three components of stress at a point defined relative to an x-y coordinate frame, then give me the angle ϕ, and I will crank out -- not only the normal and shear stress components acting on the face with its outward normal inclined at the angle ϕ with respect to the x axis, but the normal stress on the *y'* face as well. In fact I could draw a square tilted at an angle ϕ to the horizontal and show the stress components σ'_x, σ'_y and σ'_{xy} acting on the *x'* and *y'* faces.

To show the utility of these relationships consider the following scenario:

Exercise 4.3 – An solid circular cylinder made of some brittle material is subject to pure torsion —a torque M_t. If we assume that a shear stress $\tau(r)$ acts within the cylinder, distributed over any cross section, varying with r according to

$$\tau(r) = c \cdot r^n$$

where n is a positive integer, then the maximum value of τ, will occur at the outer radius of the shaft.

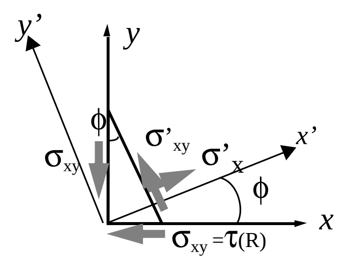
But is this the maximum value? That is, while certainly r^n is maximum at the outermost radius, $r=R$, it may very well be that the maximum shear stress acts on some other plane at that point in the cylinder.

Show that the maximum shear stress is indeed that which acts on a plane normal to the axis of the cylinder at a point on the surface of the shaft.

Show too, that the maximum normal stress in the cylinder acts

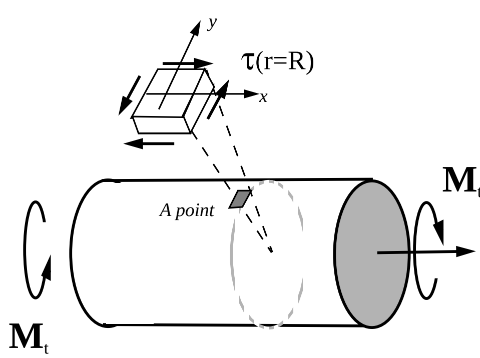
- at a point on the surface of the cylinder
- on a plane whose normal is inclined 45^o to the x axis and its value is $\sigma'_x|_{max} = \tau(R)$

We put to use our machinery for computing the stress components acting upon an arbitrarily oriented plane at a point. Our initial set of stress components for this particular state of stress is



$$\sigma_x = 0$$
$$\sigma_y = 0$$
and
$$\sigma_{xy} = \tau(R)$$

defined relative to the x-y coordinate frame shown top right. Our equations defining the transformation of components of stress at the point take the simpler form



$$\sigma'_x = \tau \cdot \sin 2\phi$$
$$\sigma'_y = -\tau \cdot \sin 2\phi$$
$$\sigma'_{xy} = \tau \cdot \cos 2\phi$$

To find the maximum value for the shear stress component with respect to the plane defined by ϕ, we set the derivative of σ'_{xy} to zero. Since there are no "boundaries" on ϕ to worry about, this ought to suffice.

So, for a maximum, we must have

$$\frac{d\sigma'_{xy}}{d\phi} = -2\tau \cdot \sin 2\phi = 0$$

Now there are many values of ϕ which satisfy this requirement, $\phi=0$, $\phi=\pi/2$, ... But all of these roots just give the orientation of the of our initial two mutually perpendicular, x-y planes. Hence the maximum shear stress within the shaft is just τ at r=R.

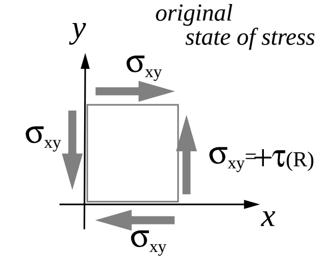
To find the extreme, including maximum, values for the normal stress, σ'_x we proceed in much the same way; differentiating our expression above for σ'_x with respect to ϕ yields

$$\frac{d\sigma'_x}{d\phi} = 2\tau \cdot \cos 2\phi = 0 \qquad \textbf{(EQ 1)}$$

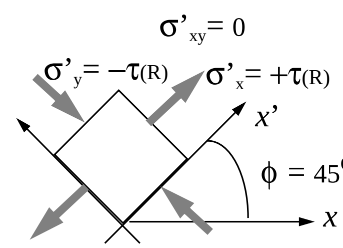
Again there is a string of values of ϕ, each of which satisfies this requirement.

We have $2\phi = \pi/2,\ 3\pi/2$ .... or $\phi = \pi/4,\ 3\pi/4$

At $\phi = \pi/4$ (= 45°), the value of the normal stress is $\sigma'_x = +\tau\, sin2\phi = \tau$. So the maximum normal stress acting at the point on the surface is equal in magnitude to the maximum shear stress component.



Note too that our transformation relations say that the normal stress component acting on the y' plane, with $\phi=\pi/4$ is negative and equal in magnitude to τ. Finally we find that the shear stress acting on the x'-y' planes is zero! We illustrate the state of stress at the point relative to the x'-y' planes below right.



Backing out of the woods in order to see the trees, we claim that if our cylinder is made of a brittle material, it will fracture across the plane upon which the maximum tensile stress acts. If you go now and take a piece of chalk and subject it to a torque until it breaks, you should see a fracture plane in the form of a helical surface inclined at 45 degrees to the axis of the cylinder. Check it out.

Of course it's not enough to know the orientation of the fracture plane when designing brittle shafts to carry torsion. We need to know the *magnitude of the torque* which will cause fracture. In other words we need to know how the shear stress does *in fact* vary throughout the cylinder.

This remains an unanswered question. So too for the beam: How do the normal stress (and shear stress) components vary over a cross section of the beam? We have claimed that to answer these questions we must go beyond the concepts and principles of static equilibrium. This we do now, looking first at simple indeterminate systems, then on to the indeterminate truss, the beam in bending and beyond.

4.5 Strain

The study of the elastic behavior of statically determinate or indeterminate truss structures serves as a paradigm for the modeling and analysis of **all** structures in so far as it illustrates

- the isolation of a region of the structure prerequisite to imagining internal forces;
- the application of the equilibrium requirements relating the internal forces to one another and to the applied forces;
- the need to consider the displacements and deformations if the structure is redundant; [1]
- and how, if displacements and deformations are introduced, then the constitution of the material(s) out of which the structure is made must be known so that the internal forces can be related to the deformations.

We are going to move on, with these items in mind, to study the elastic behavior of shafts in torsion and of beams in bending with the aim of completing the task we started in an earlier chapter – among other objectives, to determine when they might fail. To prepare for this, we step back and dig in a bit deeper to develop more complete measures of deformation, ones that are capable of taking us beyond uniaxial extension or contraction. We then must relate these measures of deformation, the *components of strain at a point* to the *components of stress at a point* through some stress-strain equations. We address that task in the next chapter.

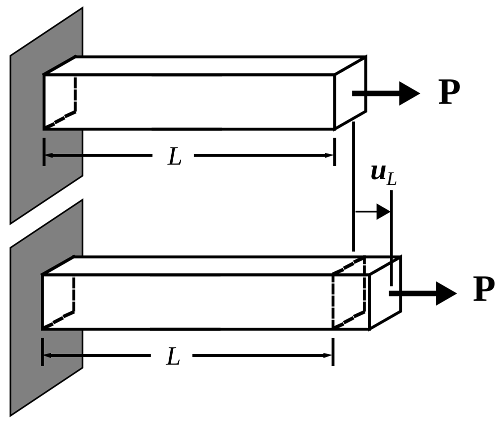
We will proceed without reference to truss members, beams, shafts in torsion, shells, membranes or whatever structural element might come to mind, but for an arbitrarily shaped body, a *continuous solid body, a solid continuum*. We put on another special pair of eyeglasses, a pair that enables us to imagine what transpires *at a point* in a solid subjected to a load which causes it to deform and engenders strain as well internal forces, now stresses. In our derivations that follow, we limit our attention to two dimensions: We first construct a set of strain measures in terms of the x,y (and z) components of displacement at a point. We then develop a set of stress/strain equations for a *linear, isotropic, homogenous, elastic solid*.

1. We of course must consider the deformations even of a determinate structure if we wish to estimate the displacements of points in the structure when loaded.

4.6 Strain at a Point

When a body displaces as a rigid body, points etched on the body will move through space but any arbitrarily chosen point will maintain the same distance from any other point–just as the stars in the sky maintain their position relative to other all other stars, night after night, as the heavens rotate about the earth Except, of course, for certain “wandering stars” which do not maintain fixed distances among themselves or from the others.

But when a body *deforms*, points move relative to one another and distances between points change. For example, when the bar shown below is pulled with an end load *P* along its axis we know that the end will displace to the right, say a distance u_L, relative to the fixed, left end of the bar.



Assuming the bar is *homogenous*, that is, its constitution does not change as we move in from the end of the bar, we anticipate that the displacement will decrease. At the wall it must be zero; at the mid point we might anticipate it will be $u_L/2$. Indeed, this was the essence of our story about the behavior of an elastic rod in a uniaxial tension test.

There we had $P = (AE/L) \cdot \delta = k \cdot \delta$

The stiffness *k* is inversely proportional to the length of the rod so that, if the *same* end load is applied to bars of different length, the displacement of the ends will be proportional to their lengths, and the ratio of δ to L will be constant.

In our mind, then, we can imagine the horizontal rod shown above cut through at its midpoint. As far as the remaining, left portion is concerned, it is fixed at its left end and sees an a load *P* at its right end. Now since it has but half the length, its *end* will displace to the right but $u_L/2$.

We can continue this thought experiment from now to eternity; each time we make a cut we will obtain a midpoint displacement which is one-half the displacement at the right end of the previously imagined section. This of course assumes the bar is uniform in its cross-sectional area and material properties — that is, the bar is *homogeneous*. We summarize this result neatly by writing

$$u(x) = (u_L/L) \cdot x$$

where the factor, (u_L/L), is a measure of the extensional strain of the bar, defined as the ratio of the change in length of the bar to its original length.

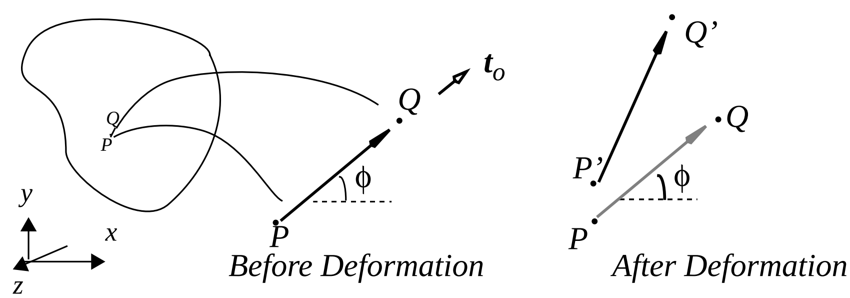
This brief thought experiment gives us a way to define a measure of extensional strain *at a point*. We say, at any point in the bar, that is, at any *x*,

$$\varepsilon_x = \lim_{\Delta x \to 0} (\Delta u / \Delta x) \Rightarrow \varepsilon_x = \frac{\partial u}{\partial x}$$

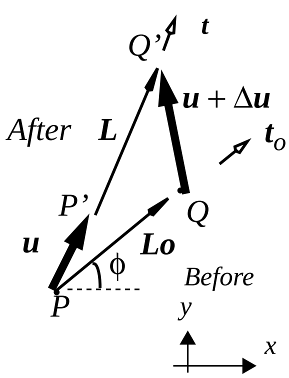
For the homogenous bar under end load *P* we see that ε_x is a constant; it does not vary with *x*. We might claim that the end displacement, u_L is *uniformly distributed* over the length; that is, the relative displacement of any two points, equidistant apart in the undeformed state, is a constant; but this is not the usual way of speaking nor, other than for a truss element, is it usually the case.

The partial derivative implies that, u, the displacement component in the *x* direction can be a function of spatial dimensions other than *x* alone, that is, for an arbitrary solid, with things changing as one moves in any of the three coordinate directions, we would have *u*= *u*(*x,y,z*). We turn to this more general situation now.

Exercise 4.4 – What do I *need to know* about the displacements of points in a solid in order to compute the extensional strain at the point *P*, arbitrarily taken, in the direction of $\boldsymbol{t}_o$, also arbitrarily chosen, as the body deforms from the state indicated at the left to that at the right?



We designate the extensional strain at *P* in the direction of $\boldsymbol{t}_0$ by ε_{PQ}. Our task is to see what we need to know in order to evaluate the limit $\varepsilon_{PQ} = \lim_{PQ \to 0} (P'Q' - PQ)/(PQ)$



To do this, we draw another picture of the undeformed and deformed differential line element, *PQ*. together with the displacements of its endpoints. Point P's displacement to P' is shown as the vector, ***u,*** while the displacement of point Q, some small distance away, is designated by ***u***+Δ***u***.

This now looks very much like the representation used in the last chapter to illustrate and construct an expression for the extension of a truss member as a function of the horizontal and vertical components of displacement at its two ends. That's why I have introduced the vectors $\boldsymbol{L}_o$, and ***L*** for the *directed line segments PQ*, *P'Q*' respectively though they are in fact meant to be small, differential lengths. Proceeding in the same way as we did in our study of the truss, we write, as a consequence of vector addition,

$$\mathbf{u} + \mathbf{L} = \mathbf{u} + \Delta\mathbf{u} + \mathbf{L_0}$$

which yields an expression for Δ ***u*** in terms of the vector difference of the two directed line segments, namely $\Delta\mathbf{u} = \mathbf{L} - \mathbf{L}_0$

We now introduce a most significant constraint, We assume, as we did with the truss, that *displacements and rotations are small* – displacements relative to some characteristic length of the solid, rotations relative to a radian. This should not to be read as implying our analysis is of limited use. Most structures behave, i.e., deform, according to this constraint and, as we have seen in our study of a truss structure, it is entirely consistent with our writing the *equilibrium equations with respect to the undeformed configuration.* In fact *not* to do so would be erroneous.

Explicitly this means we will take

$$\mathbf{t} \approx \mathbf{t_0} \qquad \text{so that} \qquad |\mathbf{L}| = \mathbf{t} \bullet \mathbf{L} \approx \mathbf{t_0} \bullet \mathbf{L}$$

With this we can claim that the change in length of the directed line segment, *PQ*, in moving to *P'Q'*, is given by the projection of $\Delta\boldsymbol{u}$ upon *PQ* that is, since

$$P'Q' - PQ = |\mathbf{L}| - |\mathbf{L_0}|$$

we have

$$P'Q' - PQ = \mathbf{t_0} \bullet \mathbf{L} - \mathbf{t}_0 \bullet \mathbf{L}_0 = \mathbf{t_0} \bullet (\mathbf{L} - \mathbf{L_0}) = \mathbf{t_0} \bullet \Delta\mathbf{u}$$

where $\boldsymbol{t}_o$ is, as before, a unit vector in the direction of *PQ*.

From here on in, constructing an expression for ε_{PQ} requires the machine-like evaluation of the scalar product, $\mathbf{t}_0 * \Delta\mathbf{u}$, the introduction of the partial derivatives of the scalar components of the displacement taken with respect to position, and the manipulation of all of this into a form which reveals what's needed in order to compute the relative change in length of the arbitrarily oriented, differential line segment, *PQ*. We work with respect to a rectangular cartesian coordinate frame, *x,y*, and define the horizontal and vertical components of the displacement vector $\boldsymbol{u}$ to be *u,v* respectively. (In the following be careful to distinguish between the scalar u and the vector **u**; the former is the x component of the latter). That is, we set

$$\mathbf{u} = u(x, y) \cdot \mathbf{i} + v(x, y)\mathbf{j}$$

where the coordinates *x,y* label the position of the point *P*. The differential change in the displacement vector in moving from *P* to *Q*, a small distance which in the limit will go to zero, may then be written

$$\Delta\mathbf{u} = \Delta u(x, y) \cdot \mathbf{i} + \Delta v(x, y) \cdot \mathbf{j}$$

Carrying out the scalar product, we obtain for the change in length of *PQ*:

$$P'Q' - PQ = \mathbf{t}_0 \bullet \Delta\mathbf{u} = (\Delta u) \cdot \cos\phi + (\Delta v) \cdot \sin\phi$$

We next approximate the small changes in the horizontal and vertical, scalar components of displacement by the products of their slopes at *P* taken with the appropriate differential lengths along the *x* and *y* axes as we move to point *Q*. That is[1]

$$\Delta u(x, y) \approx \left(\frac{\partial u}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y}\right)\Delta y$$

and

$$\Delta v(x, y) \approx \left(\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial v}{\partial y}\right)\Delta y$$

We have then

$$(P'Q' - PQ)/(PQ) \approx \left[\left(\frac{\partial u}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y}\right)\Delta y\right](\cos\phi/L_o) + \left[\left(\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial v}{\partial y}\right)\Delta y\right](\sin\phi/L_o)$$

1. It is easy to be confused in the midst of all these partial derivatives. It's worth taking five minutes to try to sort them out.

where I have introduced L_o for the original length *PQ*.

This is an approximate relationship because the changes in the horizontal and vertical components of displacement are only approximately represented by the first partial derivatives. *In the limit*, however, as the distance *PQ*, and hence as Δx, Δy approaches zero, the approximation may be made as accurate as we like. Note also, that the ratios $\Delta x/L_o$, $\Delta y/L_o$ approach $\cos\phi$ and $\sin\phi$ respectively.

We obtain, finally, letting *PQ* go to zero, the following expression for the extensional strain *at the point P* in the direction *PQ*:

$$\varepsilon_{PQ} = \left(\frac{\partial u}{\partial x}\right)\cos^2\phi + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\cos\phi\sin\phi + \left(\frac{\partial v}{\partial y}\right)\sin^2\phi$$

It appears that in order to compute ε_{PQ} in the direction ϕ we *need to know* the four first partial derivatives of the scalar components of the displacement at the point *P*. In fact, however, we do not need to know all four partial derivatives since it is enough to know the **three** bracketed terms appearing above Think of computing ε_{PQ} for different values of ϕ; knowing the values for the three bracketed terms will enable you to do this.

The relationship above is a very important piece of machinery. It tells us how to compute the extensional strain in any direction, defined by ϕ, at any point, defined by *x*,*y*, in a body. In what follows, we call the three quantities within the brackets the *three scalar components of strain at a point*. But first observe:

- If we set ϕ equal to zero in the above, which is equivalent to setting *PQ* out along the *x* axis, we obtain, as we would expect, that $\varepsilon_{PQ} = \varepsilon_x$, the extensional strain at *P* in the *x* direction, i.e.,

$$\varepsilon_x = \left(\frac{\partial u}{\partial x}\right)$$

- Our machinery is thus consistent with our previous definition of ε_x for uniaxial loading of a bar fixed at one end and lying along the x axis.
- If, in the same way, we set ϕ equal to a right angle, we obtain

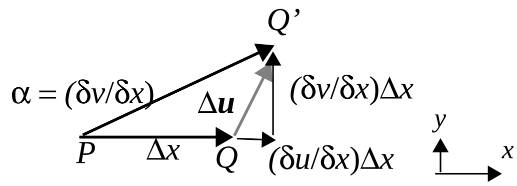
$$\varepsilon_{PQ} = \left(\frac{\partial v}{\partial y}\right)$$

which can be read as the extensional strain at *P* in the direction of a line segment along the *y* axis. We call this ε_y. That is

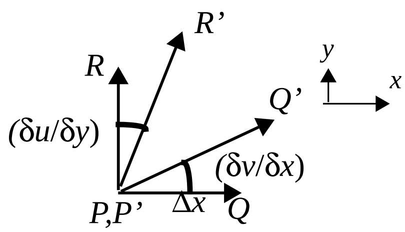
$$\varepsilon_y = \left(\frac{\partial v}{\partial y}\right)$$

- The meaning of the term $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$ is best extracted from a sketch; below we show how the term $\frac{\partial v}{\partial x}$ can be interpreted as the angle of rotation, about the *z* axis, of a line segment *PQ* along the *x* axis. For small rotations we can claim

$$\alpha \sim \tan\alpha = \frac{\left(\frac{\partial v}{\partial x}\right)\Delta x}{\Delta x}$$



Similarly, the term $\delta u/\delta y$ can be interpreted as the angle of rotation of a line segment along the y axis, but now, if positive, about the *negative z* axis. The figure below shows the meaning of both terms.



The sum of the two terms is the change in the right angle, *PQR* at point *P*. If it is a positive quantity, the right angle of the first quadrant has decreased. We define this sum to be a *shear strain* component at point *P* and label it with the symbol γ_{xy}.

- Building on the last figure, we define a *rotation* at the point *P* as the **average** of the rotations of the two, *x*,*y*, line segments. That is we define

$$\omega_{xy} = (1/2)\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Note the negative sign to account for the different directions of the two line segment rotations. If, for example, $\delta v/\delta x$ is positive, and $\delta u/\delta y = -\delta v/\delta x$ then there is *no shear strain*, no change in the right angle, but there *is* a *rotation*, of magnitude $\delta v/\delta x$ positive about the z axis at the point *P*.

These three quantities $\varepsilon_x, \gamma_{xy}, \varepsilon_y$ are the three components of strain at a point.

$$\varepsilon_x \equiv \frac{\partial u}{\partial x} \qquad \gamma_{xy} \equiv \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \qquad \varepsilon_y \equiv \frac{\partial v}{\partial y}$$

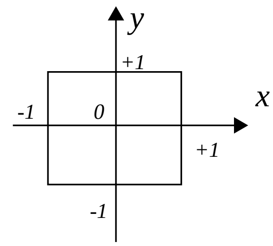
If we know the way $\varepsilon_x(x,y)$, $\gamma_{xy}(x,y)$, and $\varepsilon_y(x,y)$ vary, we say we know the *state of strain* at any point in the body. We can then write our equation for computing the extensional strain in any arbitrary direction in terms of these three strain components associated with the *x*,*y* frame at a point as:

$$\varepsilon_{PQ} = \varepsilon_x \cdot \cos\phi^2 + \gamma_{xy} \cdot \cos\phi\sin\phi + \varepsilon_y \cdot \sin\phi^2$$

Finally, note that if we are given the displacement components as continuous functions *x and y* we can, by taking the appropriate partial derivatives, compute a set of strain functions, also continuous in *x*,*y*. On the other hand, going the other way, given the three strain components, ε_x, γ_{xy}, ε_y as continuous functions of position, we cannot be assured that we can determine unique, continuous functions for the two displacement components from an integration of the strain-displacement relations. We say that the strains represent a *compatible* state of deformation only if we can do so, that is, only if we can construct a continuous *displacement field* from the strain components.

Exercise 4.5–For the planar displacement field defined by

$$u(x, y) = -\kappa \cdot xy \qquad\qquad v(x, y) = \kappa \cdot (x^2/2)$$



where κ = 0.25, sketch the locus of the edges of a 2x2 square, centered at the origin, after deformation and construct expressions for the strain components ε_x, ε_y, and γ_{xy}

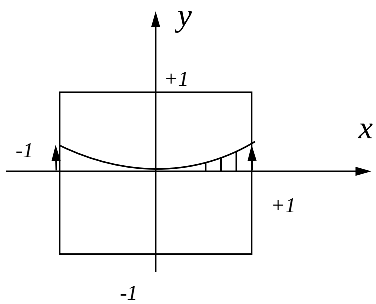
We start by evaluating the components of strain; we obtain

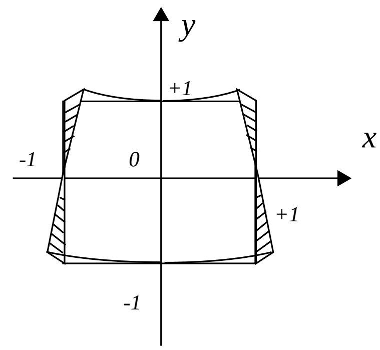
$$\varepsilon_x \equiv \frac{\partial u}{\partial x} = -\kappa y \qquad \gamma_{xy} \equiv \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = -\kappa x + \kappa y = 0 \qquad \varepsilon_y \equiv \frac{\partial v}{\partial y} = 0$$

We see that the only non zero strain is the extensional strain in the *x* direction at every point in the plane. In particular, right angles formed by the intersection of a line segment in the *x* direction with another in the *y* direction remain right angles since the shear strain vanishes. The average rotation of these intersecting line segments at each and every point is found to be

$$\omega_{xy} = (1/2)\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \kappa x$$

We sketch the locus of selected points and line segments below:





Focus first, on the figure at the left above which shows the deformed position of the points that originally lay along the *x* axis, at *y*=0. The vertical component of displacement *v* describes a parabola in the deformed state. Furthermore, the points along the *x* axis experience no horizontal displacement.

On the other hand, the points off the *x* or the *y* axis all have a horizontal component of displacement - as well as vertical. Consider now the figure above right. For example the point (1,1) moves to the left a distance *0.25* while moving up a distance *0.125*. Below the *x* axis, however, the point originally at (1,-1) moves to the right *0.25* while it still displaces upward the same *0.125*. The shaded lines are meant to indicate the ***u*** at each point.

Observe

- The state of strain does not vary with *x*, but does so with *y*.
- Right angles formed by *x-y* line segments remain right angles, that is the *shear strain* is zero.
- The average rotations of these right angles *does* vary with *x* but not with *y*. Note too that we have seemingly violated the assumption of small rotations. We did so in order to better illustrate the deformed pattern.

4.7 Transformation of Components of Strain

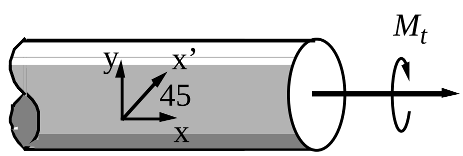
The axial stress in a truss member is related to the extensional strain in the member through an equation that looks very much like that which relates the force in a spring to its deflection. We shall relate all stress and strain components through some more general constitutive relations — equations which bring the specific properties of the material into the picture. But stress and strain are “relations” in another sense, in a more abstract, mathematical way: They are both the **same kind of mathematical entity**. The criterion and basis for this claim is the following: **The components of stress and strain at a point transform according to the same relations.** By transform we mean change; by change we mean change due to a rotation of our reference axis at the point.

Our study of how the components of strain and stress transform is motivated as much by the usefulness of this knowledge in engineering practice as by visions of mathematical elegance and sophistication. For, although this section could have been labeled *the transformation of symmetric, second-order tensors*, we have already seen an example, back in our study of stress, an example suggesting the potential utility of the component transformation machinery. We do an exercise very similar to that we tackled before to refresh our memory.

Exercise 4.6 – Three *strain gages*, attached to the surface of a solid shaft in torsion in the directions ***x, y***, and *x’* measure the three extensional strains

$$\varepsilon_x = 0 \qquad \varepsilon_y = 0 \qquad \text{and} \qquad \varepsilon_{x'} = 0.00032$$

Estimate the shear strain γ_{xy}.



Let’s work backwards. No one says you have to work forward from the “givens” straight through to the answer[1]. We are given the values of three extensional strains measured at a point on the surface of the shaft. It’s not really a point but a region about the size of a small coin.

The task is to determine the shear strain at the point from the three, measured extensional strains.

From the previous section we know that the extensional strain in the x’ direction - thinking of that direction as "PQ" - can be expressed as

$$\varepsilon_{PQ} = \varepsilon_x \cdot \cos\phi^2 + \gamma_{xy} \cdot \cos\phi\sin\phi + \varepsilon_y \cdot \sin\phi^2 ,$$

which tells me how to compute the extensional strain in some arbitrarily oriented direction at a point, as defined by the angle ϕ, given the state of strain at the point as defined by the three components of strain with respect to an *x*,*y* axis.

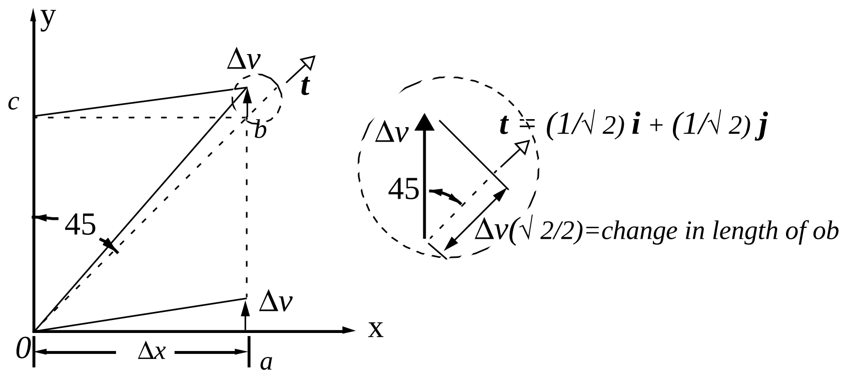
1. This is characteristic of most work, not only in engineering but in science as well. The desired end state – the answer to the problem, the basic form of a design, the theorem to be proven, the character of the data to be collected – is usually known at the outset. There are really very few surprises in science or engineering in this respect. What **is** surprising, and exciting, and rewarding is that you can manage to construct things to come out right and they work according to your expectations.

But, working backwards, I will use this to compute the shear strain γ_{xy} given knowledge of the extensional strain ε_{PQ} where *PQ* is read as the direction of the gage *x'* oriented at 45^o to the axis of the shaft and pasted to its surface. Now both ε_x and ε_y are zero[1] so this equation gives

$$0.000032 = \gamma_{xy} \cdot (1/2) \qquad \text{or} \qquad \gamma_{xy} = 0.00064$$

Observe:

- If the strains ε_x and ε_y were different from zero we would still use this relationship to obtain an estimate of the shear strain. The former would provide us with direct estimates of any axial or hoop strain.
- I can graphically interpret this equation for determining the shear strain by constructing a compatible (continuous) displacement field from the strain components ε_x, ε_y, and γ_{xy}. Note this is not the only displacement field I might generate that is consistent with these strain components but it will serve to illustrate the relationship.



With the shaft oriented horizontally and twisted as shown, I take the displacement component, u(x,y) to be zero and v(x,y) to be proportional to x but independent of y. Then the points *a* and *b* both displace vertically a distance Δv with respect to points *0* and *c*. The extension of the diagonal *0b* is, for small displacements and rotations, the projection of Δv at *b* upon the diagonal itself. So the change in length is given by $\Delta v\ (1/\sqrt{2})$. Its original length is $\sqrt{2}\Delta x$ so we can write

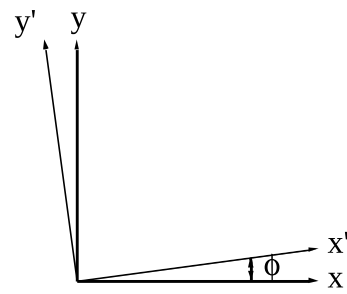
$$\varepsilon_{x'} = \varepsilon_{ob} = \frac{(\Delta v/\Delta x)}{2}$$

But, again for small rotations, $\Delta v/\Delta x = \gamma_{xy}$ the decrease in the right angle, the shear strain. Thus, as before,

$$\varepsilon_{x'} = \gamma_{xy} \cdot (1/2)$$

1. More realistic values would be some small, insignificant numbers due to *noise* or slight imbalance in the apparatus used to measure, condition, and amplify the signal produced by the strain gage. Even so, if the shaft was subject to forms of loading other than, and in addition to the torque we seek to estimate, and these engendered significant strains in the *a* and *c* directions we would still make use of this relationship in estimating the shear strain.

This exercise illustrates an application of the rules governing the transformation of the components of strain at a point. That's now the way we read the equation we in the previous section – as a way to obtain the **extensional strain** along one axis of an arbitrarily oriented coordinate frame at a point in terms of the strain components known with respect to some reference coordinate frame.



For example, if I let the arbitrarily oriented frame be labeled *x'-y'*, then the **extensional** strain components relative to this new axis system can be written in terms of the strain components associated with the original, *x-y* frame as

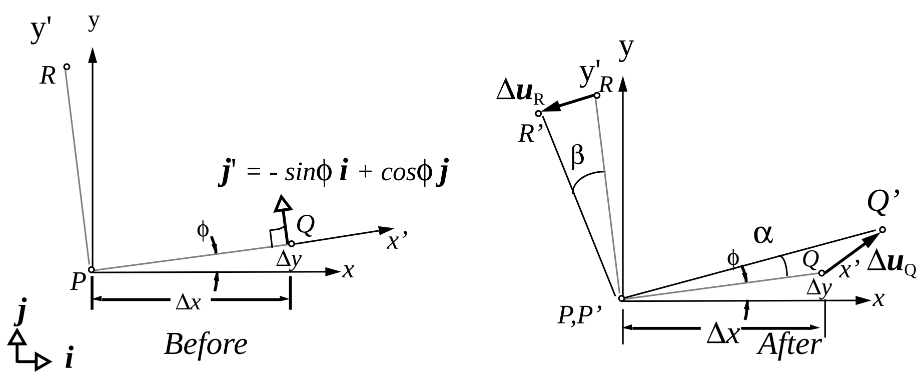
$$\varepsilon'_x = \varepsilon_x \cdot \cos\phi^2 + \gamma_{xy} \cdot \cos\phi \sin\phi + \varepsilon_y \cdot \sin\phi^2$$

$$\varepsilon'_y = \varepsilon_x \cdot \cos\phi^2 - \gamma_{xy} \cdot \cos\phi \sin\phi + \varepsilon_y \cdot \sin\phi^2$$

In obtaining the expression for the extensional strain in the *y*' direction, I substituted $\phi + \pi/2$ for ϕ in the first equation.

But there is more to the story. I must construct an equation that allows me to compute the **shear strain**, γ_{xy}' relative to the arbitrarily oriented frame, x'y'. To do so I make use of the same graphical methods of the previous section.

The figure below left shows the orientation of my reference *x-y* axis and the orientation of an arbitrarily oriented frame x'-y'. *PQ* is a differential line element in the undeformed state lying along the x' axis. ***t*** is a unit vector along *PQ*; ***e*** is a unit vector perpendicular to *PQ* in the sense shown. Δx, Δy are the horizontal and vertical coordinates of *Q* relative to the origin of the reference frame.



On the right we show the position of *PQ* in the deformed state as *P'Q*'. The displacement of point *Q relative to P* is shown as $\Delta \boldsymbol{u}_Q$. The angle α is the (small) rotation of the line element *PQ*. This is what we seek to express in terms of the strain components ε_x, ε_y *and* γ_{xy} at the point. We will also determine the rotation of a line element along the *y*' axis. Knowing these we can compute the change in the right angle *QPR* , the shear strain component with respect to the *x'-y'* system which we will mark with a "prime", γ_{xy}'.

The angle α is given approximately by $\alpha = \Delta\mathbf{u} \bullet \mathbf{j}'/(PQ)$ where ***j'*** is perpendicular to PQ.

The displacement vector we write as $\Delta\mathbf{u} = \Delta u \cdot \mathbf{i} + \Delta v \cdot \mathbf{j}$ which, to first order may be written in terms of the partial derivatives of the scalar components of the relative displacement of Q.

$$\Delta\mathbf{u} = \left(\Delta x\frac{\partial u}{\partial x} + \Delta y\frac{\partial u}{\partial y}\right)\mathbf{i} + \left(\Delta x\frac{\partial v}{\partial x} + \Delta y\frac{\partial v}{\partial y}\right)\mathbf{j}$$

and the unit vector is $\mathbf{j}' = -\sin\phi \cdot \mathbf{i} + \cos\phi \cdot \mathbf{j}$

Carrying out the scalar, dot product, noting that

$$\Delta y / PQ = \cos\phi \qquad \text{and} \qquad \Delta y / PQ = \sin\phi$$

we obtain

$$\alpha \approx -\sin\phi\left(\cos\phi\frac{\partial u}{\partial x} + \sin\phi\frac{\partial u}{\partial y}\right) + \cos\phi\left(\cos\phi\frac{\partial v}{\partial x} + \sin\phi\frac{\partial v}{\partial y}\right)$$

Or collecting terms

$$\alpha \approx \sin\phi\cos\phi\left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right) + \cos^2\phi\frac{\partial v}{\partial x} - \sin^2\phi\frac{\partial u}{\partial y}$$

I obtain the angle β the rotation of a line segment *PR* originally oriented along the *y*' axis most simply by letting ϕ go to $\phi + \pi/2$ in the above equation for the angle α. Thus

$$\beta \approx -\sin\phi\cos\phi\left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right) + \sin^2\phi\frac{\partial v}{\partial x} - \cos^2\phi\frac{\partial u}{\partial y}$$

the diminution in the right angle QPR is just α - β so I obtain:

$$\gamma'_{xy} \approx 2\sin\phi\cos\phi\left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right) + (\cos^2\phi - \sin^2\phi)\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

which, in terms of the strain components associated with the *x*,*y* axes becomes

$$\gamma'_{xy} = 2(\varepsilon_y - \varepsilon_x) \cdot \sin\phi\cos\phi + \gamma_{xy} \cdot (\cos\phi^2 - \sin\phi^2)$$

With this I have all the machinery I need to compute the components of strain with respect to one orientation of axes at a point **given** their values with respect to another. I summarize below, making use of the *double angle* identities for the $\cos\phi$ and the $\sin\phi$, namely, $\cos 2\phi = \cos^2\phi - \sin^2\phi$ and $\sin 2\phi = 2\sin\phi\cos\phi$.

$$\varepsilon'_x = \left[\frac{(\varepsilon_x + \varepsilon_y)}{2}\right] + \left[\frac{(\varepsilon_x - \varepsilon_y)}{2}\right] \cdot \cos 2\phi + (\gamma_{xy}/2)\sin 2\phi$$

$$\varepsilon'_y = \left[\frac{(\varepsilon_x + \varepsilon_y)}{2}\right] - \left[\frac{(\varepsilon_x - \varepsilon_y)}{2}\right] \cdot \cos 2\phi - (\gamma_{xy}/2)\sin 2\phi$$

$$(\gamma'_{xy}/2) = -\left[\frac{(\varepsilon_x - \varepsilon_y)}{2}\right] \cdot \sin 2\phi + (\gamma_{xy}/2)\cos 2\phi$$

I have introduced a common factor of (1/2) in the equation for the shear strain for the following reasons: If you compare these transformation relationships with those we derived for the components of stress, back in section 3.6, you will see they are identical in form if we identify the normal strain components with their corresponding normal stress components but we must identify τ_{xy} with $\gamma_{xy}/2$.

One additional relationship about deformation follows from our analysis: If I average the angular rotations of the two orthogonal line segments *PQ* and *PR*, I obtain an expression for what we define as the *rotation* of the *x'-y'* axes at the point. This produces

$$\omega'_{xy} = \left(\frac{1}{2}\right)(\alpha + \beta) = \left(\frac{1}{2}\right)\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \omega_{xy}$$

This, we note, is identical to ω_{xy} which is what justifies labeling this measure of deformation a *rigid body rotation*. It is also *invariant* of the transformation; regardless of the orientation of the coordinate frame at the point, you will always get the same number for this measure of rotation.

Exercise 4.7 – A “bug” in my graphics software distorts the image appearing on my monitor. Horizontal lines are stretched 1%; vertical lines are compressed 5% and there is a distortion of the right angles formed by the intersection of horizontal and vertical lines of approximately 3^o – a decrease in right angle in the first quadrant. *Estimate* the maximum extensional distortion I can anticipate for an arbitrarily oriented line drawn by my software. What is the orientation of this particular line relative to the horizontal?

I seek a maximum value for the extensional strain at a point — the extensional strain of an arbitrarily oriented line segment which is maximum. Any point on the screen will serve; we are working with a *homogeneous* state of strain, one which does not vary with position. I also of course want to know the direction of this line segment. The equation above for ε_x' shows the extensional strain as a function of ϕ; we differentiate with respect to ϕ seeking the value for the angle which will give a maximum (or minimum) extensional strain. I have:

$$\frac{d\varepsilon'_x}{d\phi} = -(\varepsilon_y - \varepsilon_x)\sin 2\phi + \gamma_{xy} \cdot \cos 2\phi = 0$$

which I manipulate to

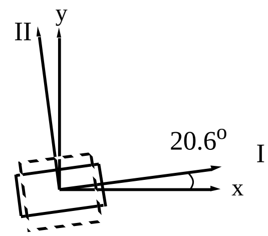
$$\tan 2\phi = \gamma_{xy} / (\varepsilon_y - \varepsilon_x)$$

Now the three *x*,*y* components of strain are ε_x=0.01, ε_y = -0.05, *and* γ_{xy} = 3/57.3 = 0.052. The above relationship, because of the behavior of the tangent function, will give me two roots within the range 0 < ϕ < 360^o, hence two values of ϕ.

I obtain two possibilities for the angle of orientation of maximum (or minimum) extensional strain, $\phi = 20.6^o$ *and* $\phi = 20.6 + 90^o = 110.6^o$ One of these will correspond to a maximum extensional strain, the other to a minimum. Note that we can read the second root as an extensional strain in a direction perpendicular to that associated with the first root. In other words, if we evaluate both ε_x' and ε_y'for a rotation of ϕ = 20.6^o we will find one a maximum the other a minimum. This we do now.

Taking then, $\phi = 20.6^o$ I obtain for the extensional strain in that direction, $\varepsilon_I = 0.0197$

about two percent extension. The extensional strain at right angles to this I obtain from the equation for ε_y', a strain along an axis 110.6^o around from the horizontal, $\varepsilon_{II} = -0.0597$, about six percent contraction. This latter is the maximum extensional distortion, a contraction of 5.97%. We illustrate the situation below.



Observe

- We call this pair of extreme values of extensional strain at a point, one a maximum, the other a minimum, *the principal strains*; the axes they are associated with are called *the principal axes*.
- The **shear strain associated with the principal axes is zero, always**. This follows from comparing the equation we derived by setting the derivative of the arbitrarily oriented extensional strain with respect to angle of rotation, namely

$$\tan(2\phi) = \gamma_{xy}/(\varepsilon_x - \varepsilon_y)$$

with the equation for the transformed component γ_{xy}'. If the former is satisfied then the shear must vanish.

4.8 Mohr's Circle

Our working up of the transformation relations for stress and for strain and our exploration of their meaning in terms of extreme values has required considerable mathematical manipulation. We turn now to a graphical rendering of these relationships called *Mohr's Circle*[1]. I will set out the rules for constructing the circle for a particular state of stress, (note that we could just as well use strain as a vehicle for this exposition), show how to read the pattern, then comment about its legitimacy.

First, I repeat the transformation equations for a two-dimensional state of stress. They look just like the transformation relations for strain, which we recently derived, if we replace half the shear strain with the corresponding shear stress:

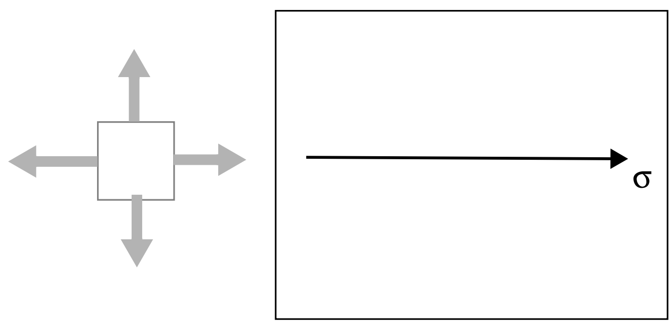
$$\sigma'_x = \left[\frac{(\sigma_x + \sigma_y)}{2}\right] + \left[\frac{(\sigma_x - \sigma_y)}{2}\right] \cdot \cos 2\phi + \sigma_{xy} \sin 2\phi$$

$$\sigma'_y = \left[\frac{(\sigma_x + \sigma_y)}{2}\right] - \left[\frac{(\sigma_x - \sigma_y)}{2}\right] \cdot \cos 2\phi - \sigma_{xy} \sin 2\phi$$

$$\sigma'_{xy} = -\left[\frac{(\sigma_x - \sigma_y)}{2}\right] \cdot \sin 2\phi + \sigma_{xy} \cos 2\phi$$

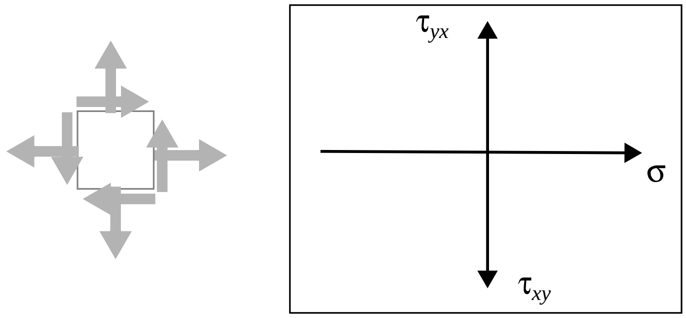
To construct Mohr's Circle, given the state of stress $\sigma_x = 7$, $\tau_{xy} = 4$, *and* $\sigma_y = 1$ we proceed as follows: Note that I have dropped all pretense of reality in this choice of values for the components of stress. As we shall see, it is their relative magnitudes that is important to this geometric construction. Everything will scale by any common factor you please to apply. You could think of these as $\sigma_x = 7 \times 10^3$ KN/m^2 ...etc., if you like.

1. Lay out a horizontal axis and label it σ positive to the right.

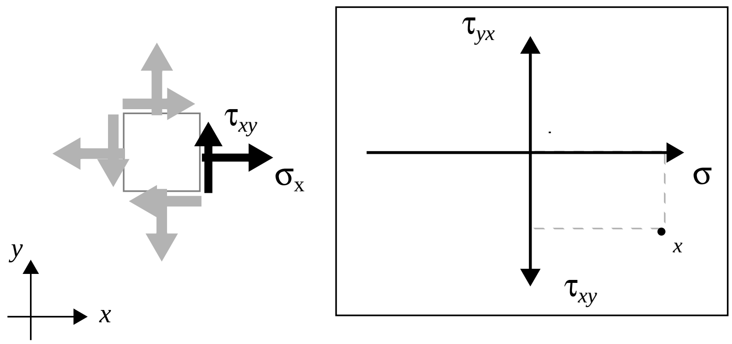


1. Studying Mohr's Circle is customarily the final act in this first stage of indoctrination into Engineering Mechanics. Your uninitiated colleagues may be able to grasp the concepts of a truss member in tension or compression, a beam in bending, a shaft in torsion using their common sense knowledge of the world around them and some prompting from you, but Mohr's Circle will appear as a complete mystery, an unfathomable ritual of signs, circles, and greek symbols. Although it does not tell us anything new, over and above all that we have done up to this point in the chapter, once you've mastered the technique it will set you apart from the crowd and shape your very well being. It may also provide you with a useful aid to understanding the transformation of stress and strain at a point on occasion.

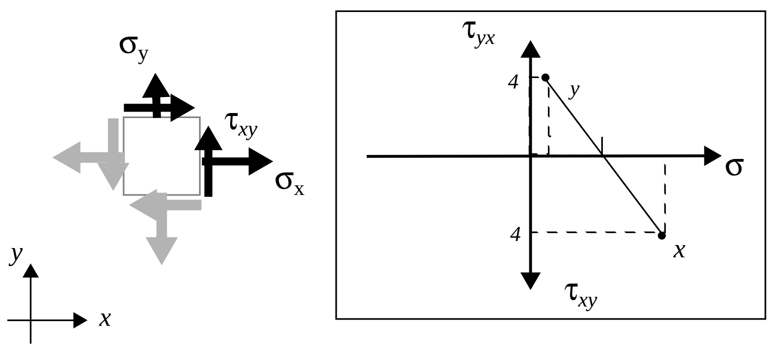
2. Lay out an axis perpendicular to the above and label it τ_{xy} **positive down** and τ_{yx} **positive up**.



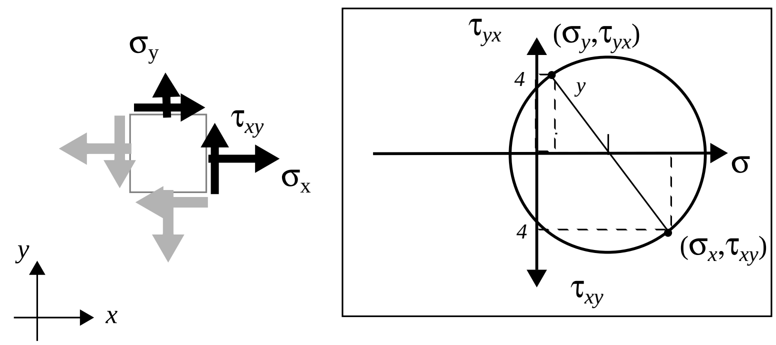
3. Plot a point associated with the stress components acting on an *x* face at the coordinates $(\sigma_x,\tau_{xy})=(7,4_{down})$. Label it x_{face}, or *x* if you are cramped for space.



4. Plot a second point associated with the stress components acting on an *y* face at the coordinates $(\sigma_y,\tau_{yx})= (1,4_{up})$. Label it y_{face}, or *y* if you are cramped for space. Connect the two points with a straight line. Note the order of the subscripts on the shear stress..



5. Chanting "similar triangles", note that the center of the line must necessarily lie on the horizontal, σ axis since $\tau_{xy}=\tau_{yx}$, 4=4. Draw a circle with the line as a diagonal.

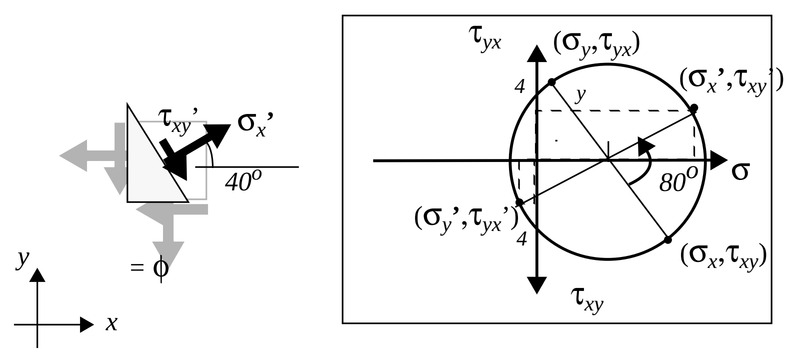


6. Note that the radius of this circle is

$$R_{Mohrs} = \sqrt{(\tau_{xy})^2 + [(\sigma_x - \sigma_y)/2]^2}$$

which for the numbers we are using is just $R_{Mohr's\ C} = 5$, and its center lies at $(\sigma_x+\sigma_y)/2 = 4$.

7. To find the stress components acting on a plane whose normal is inclined at an angle of ϕ degrees, positive counterclockwise, to the *x* axis **in the physical plane**, rotate the diagonal 2ϕ **in the Mohr's Circle plane**. We illustrate this for $\phi = 40^o$. Note that the shear stress on the new *x'* face is negative according to the convention we have chosen for our Mohr's Circle.[1]



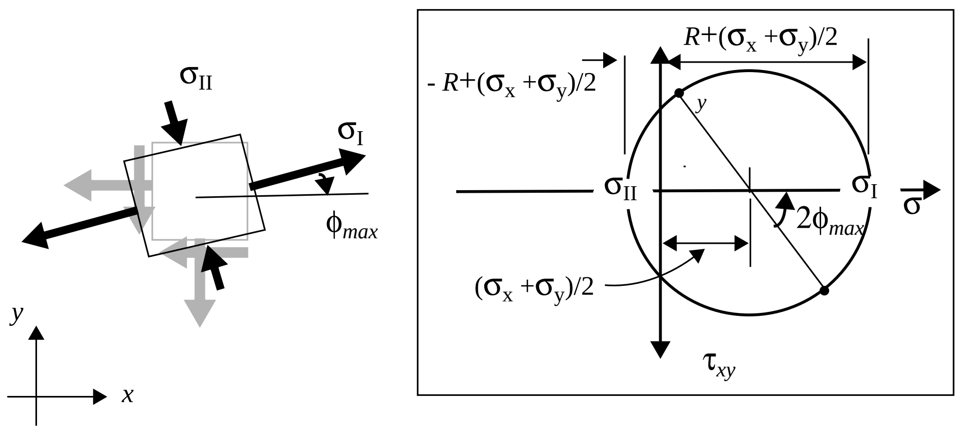
1. Warning: Other texts use other conventions.

8. The stress components acting on the *y*’ face, at $\phi + \pi/2 = 130^o$ around in the **physical plane** are $2\phi + \pi = 240^o$ around in the **Mohr’s Circle plane**, just 2ϕ around from the *y* face in the Mohr’s Circle plane.

We establish the legitimacy of this graphical representation of the transformation equations for stress making the following observations:

- The extreme values of the normal stress lie at the two intersections of the circle with the σ axis. The angle of rotation from the x_{face} to the principal plane *I* on the Mohr’s Circle is related to the stress components by the equation previously derived:

$$\tan 2\phi = 2\tau_{xy}/(\sigma_x - \sigma_y). \qquad \textbf{(EQ 2)}$$



- Note that on the principal planes the shear stress vanishes.

- The values of the two principal stresses can be written in terms of the radius of the circle.

$$\sigma_{I,\,II} = [(\sigma_x + \sigma_y)/2] \pm \sqrt{(\tau_{xy})^2 + [(\sigma_x - \sigma_y)/2]^2}$$

- The orientation of the planes upon which an extreme value for the shear stress acts is obtained from a rotation of 90^o around from the σ axis on the Mohr’s Circle. The corresponding rotation in the physical plane is 45^o.

- The sum $\sigma_x + \sigma_y$ is an invariant of the transformation. The center of the Mohr’s Circle does not move. This result too can be obtained from the equations derived simply by adding the expression for σ_x’ to that obtained for σ_y’.

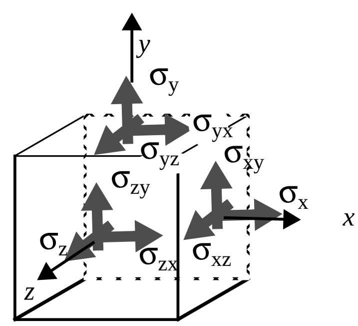
- So too the radius of the Mohr’s Circle is an invariant. This takes a little more effort to prove.

4.9 Stress/Strain Relations

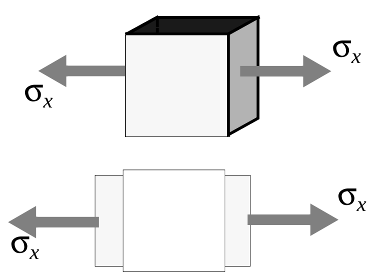
We now want to develop a set of stress/strain relations for a continuous body, equations which apply at each and every point throughout the continuum. In this we will restrict our attention, at least in this chapter, to certain type of materials namely *homogeneous, linear, elastic, isotropic bodies*.

- *Homogeneous* means that the properties of the body do not vary from one point in the body to another.
- *Linear* means that the equations relating stress and strain are linear; changes in stress are directly proportional to changes in strain (and the other way around, too).
- *Elastic* means that the body returns to its original, undeformed configuration when the applied forces and/or moments are removed.
- *Isotropic* means that the stress strain relations do not change with direction at a point. This means that a laminated material, a material with a preferred orientation of "grains" at the microscopic level, are outside our field of view, at least for the moment.

We have talked about *stress at a point*. We drew a figure like the one at the right to help us visualize the nature of the normal and shear components of stress at a point. We say the *state of stress is fully specified* by the normal components, σ_x, σ_y σ_z and the shear components $\sigma_{xy}=\sigma_{yx}$, $\sigma_{yz}=\sigma_{zy}$ $\sigma_{xz}=\sigma_{zx}$.



With these restrictions and a heavy dose of symmetry, we will be able to construct a set of stress/strain equations that will apply to many structural materials. This we do now, performing a sequence of thought experiments in which we apply to an element of stuff at a point each stress component in turn and imagine what strains will be engendered, which ones will not. Again, symmetry will be crucial to our constructions.



We start by applying the normal stress component σ_x alone.

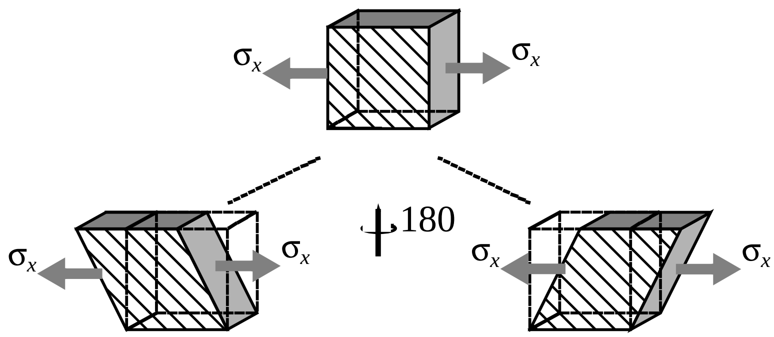
We expect to see some extensional strain ε_x. We might not anticipate normal strains in the other two coordinate directions but there is nothing to rule them out, so we posit an ε_y and an ε_z.

Now these, because of the indifference of the material to the orientation of the *y* and the *z* axis,– that is, from symmetry– these must be equal. We can say nothing more on the basis of our symmetrical thoughts alone. At this point, we will cheat, and introduce a real piece of experimental data, namely that the material *contracts* in the *y* and *z* directions as it extends in the *x* direction due to the applied σ_x. We write then, for the strains due to a σ_x:

$$\varepsilon_x = \sigma_x / E \qquad \varepsilon_y = \varepsilon_z = -\upsilon\varepsilon_x$$

In these we have made use of another bit of real experimental evidence in designating the constant of proportionality in the relationship between the extensional strain in the direction of the applied normal stress to be the *elastic, or Young's modulus, E*. The ratio of the lateral contraction to the extension, the so called *Poisson's ratio* is designated by the symbol ν. We have encountered the magnitude of the elastic modulus E for 1020, cold rolled steel in the previous chapter. Poisson's ratio, ν is new; it takes on values on the order of one-quarter to one-half, the latter value characterizing an[1] incompressible material.

But what about the shear strains? Does σ_x engender any shear strains? The answer is no and here symmetry is all that we need to reach this conclusion. The sketch below shows two possible configurations for the shear strain γ_{xy}. Both are equally possible to an unbiased observer. But which one will follow the application of σ_x?



There is no reason why one or the other should occur.[2] Indeed they are in contradiction to one another; that is, if you say the one at the left occurs, I, by running around to the other side of the page, or more easily, by imagining the bit on the left rotated 180^o about a vertical axis, can obtain the configuration at the right. But this is impossible. These two dramatically different configurations cannot exist at the same time. Hence, neither of them is a possibility; a normal stress σ_x will not induce a γ_{xy}, or for that matter, a γ_{xz} shear strain.By symmetry again, we can rule out the possibility of a γ_{yz}. We conclude, then, that under the action of the stress component σ_x alone, we obtain the extensional strains written out above.

Our next step is to apply a stress component σ_y alone. But since the body is isotropic, it does not differentiate between the x and y directions. Hence our task is easy; we simply replace x by y (and y by x) in the above relationships and we have that, under the action of the stress component σ_y alone, we obtain the extensional strains

$$\varepsilon_y = \sigma_y / E \qquad\qquad \varepsilon_x = \varepsilon_z = -\upsilon(\sigma_y / E)$$

The same argument applies when we apply the normal stress σ_z alone.

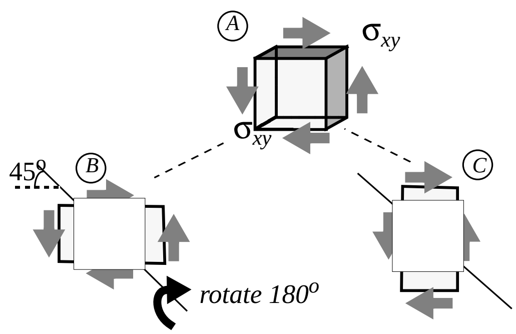
Now if we apply all three components of normal stress together, we will generate the extensional strains, and only the extensional strains,

1. In fact, Poisson proved that, for an isotropic body, Poisson's ratio should be exactly one-quarter. We claim today that he was working with a faulty model of the continuum. For some relevant history on early nineteenth century developments in the continuum theories see Bucciarelli and Dworsky, SOPHIE GERMAIN, an Essay in the Development of the Theory of Elasticity

2. Think of the icon at the top as Buridan's ass, the two below as bales of hay.

$$\varepsilon_x = (1/E) \cdot [\sigma_x - \nu(\sigma_y + \sigma_z)]$$
$$\varepsilon_y = (1/E) \cdot [\sigma_y - \nu(\sigma_x + \sigma_z)]$$
and
$$\varepsilon_z = (1/E) \cdot [\sigma_z - \nu(\sigma_x + \sigma_y)]$$

One possibility remains: What if we apply a shear **stress**? Will this produce an extensional strain component in any of the three coordinate directions? The answer is no, and symmetry again rules. For example, say we apply a shear stress, σ_{xy}. The figure below shows two possible, shortly to be shown impossible, geometries of deformation which include extensional straining.



Now I imagine rotating the one on the left about an axis inclined at 45^o as indicated. I produce the configuration on the right. Try this with a piece of rectangular paper, a 3 by 5 card, or the like. But this is an impossible situation. The two configurations are mutually contradictory. A like cause, in this case a positive shear stress at the point, should produce a like effect. This is not the case. Hence, neither the deformation of *B* nor of *C* is possible.

There remains one further possibility: that a σ_{xy} generates an extensional strain in the *x* direction equal to that in the *y* direction. But this too can be ruled out by symmetry. We conclude then that the shear strain σ_{xy}, or σ_{yz} or σ_{xz} for that matter, produces no extensional strains.

The expressions for the extensional strains above are not quite complete. We take the opportunity at this point to introduce another quite distinct cause of the deformation of solids, namely a *temperature change*. The effect of a temperature change is to produce an extensional strain proportional to the change. That is, for an isotropic body,

$$\varepsilon_{x \text{ or } y \text{ or } z} = \alpha \Delta T$$

The *coefficient of thermal expansion*, α, has units of $1/^oC$ or $1/^oF$ and for most structural materials is a positive quantity on the order of 10^{-6}. Materials with a *negative coefficient of expansion* deserve to be labeled *exotic*. They are few and far between.

The equations for the extensional components of strain in terms of stress and temperature change then can be written

In the above, we ruled out the possibility of a shear stress producing an extensional strain. A shear stress produces, as you might expect, a shear strain. We state without demonstration that a shear stress produces only the corresponding shear strain. Furthermore, a temperature change induces no shear strain at a point. The remaining three equations relating the components of stress at a point in a linear, elastic, isotropic body are then.

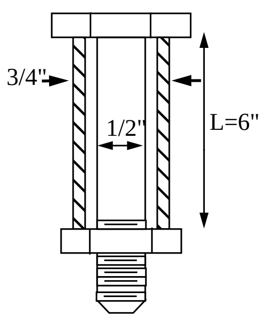
$$\varepsilon_x = (1/E) \cdot [\sigma_x - \nu(\sigma_y + \sigma_z)] + \alpha\Delta T$$
$$\varepsilon_y = (1/E) \cdot [\sigma_y - \nu(\sigma_x + \sigma_z)] + \alpha\Delta T$$
and
$$\varepsilon_z = (1/E) \cdot [\sigma_z - \nu(\sigma_x + \sigma_y)] + \alpha\Delta T$$

$$\gamma_{xy} = \sigma_{xy}/G$$
$$\gamma_{xz} = \sigma_{xz}/G$$
and
$$\gamma_{yz} = \sigma_{yz}/G$$

Recall that $\sigma_{xy} = \sigma_{yx}$. In these, *G, the shear modulus* is apparently a third elastic constant but we shall show in time that *G* can be expressed in terms of the elastic modulus and Poisson's ratio according to:

$$G = \frac{E}{2(1 + \upsilon)}$$

Exercise 4.8 A steel bolt, of 1/2 inch diameter, is surrounded by an aluminum cylindrical sleeve of 3/4" diameter and wall thickness, *t*= 0.10 in. The bolt has 32 threads/inch and when the material is at a temperature of 40^oC the nut is tightened one-quarter turn.



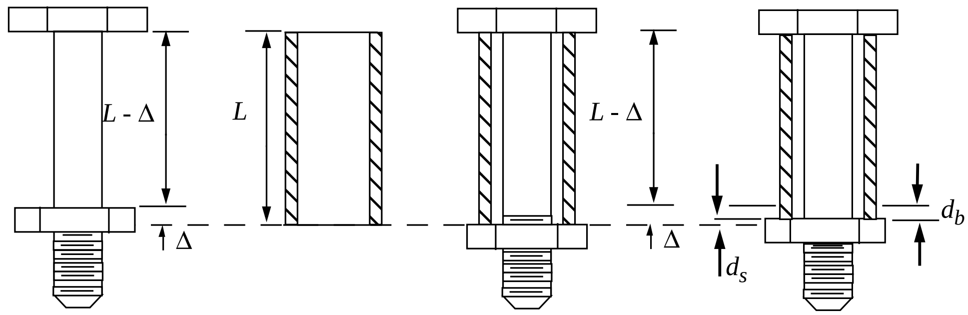
Show that the uniaxial stresses acting in the bolt and in the sleeve at this temperature are $\sigma_{bolt} = 79\ MN/m^2$, *and* $\sigma_{sleeve} = -63\ MN/m^2$

where the negative sign indicates the aluminum sleeve is in compression. *What if* the bolt and nut are cooled; at what temperature might the bolt become loose in the sleeve?

Compatibility of Deformation

Compatibility of Deformation is best assured by playing out a thought experiment about how the bolt and sleeve go from their initial unstressed, undeformed state to the final state. Think of the bolt and nut being separate from the sleeve.

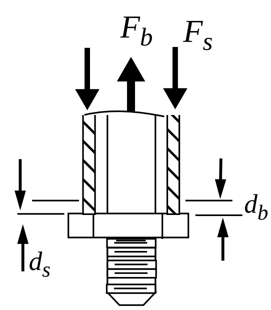
Think then of turning down the nut one quarter turn. We show this state at the left. Δ is the distance traveled in one quarter turn which, at 1/32 *inch/turn* is just $\Delta = 1/128$ *in*



Next think of stretching the bolt out until we can once again fit the nut-bolt over the aluminum sleeve, the latter still in its undeformed state. This is shown in the middle figure above. Now, while stretching out the bolt in this way, place over the aluminum sleeve[1]. The bolt will strive to return to its undeformed length – the behavior is assumed to be elastic – while the aluminum sleeve will resist contraction. The final state is shown at the right. The net result is that the steel bolt has extended **from its undeformed state** a distance $d_b$while the aluminum sleeve has contracted a distance d_s. We see from the geometry of these three figures that we must have, for compatibility of deformation, $d_s + d_b = \Delta$ which is one equation in two unknowns.

Equilibrium

The figure below shows an isolation made by cutting through the bolt and the sleeve at some arbitrary section along the axis. **Note in this I have violated my usual convention. I have taken the force in the aluminum as positive in compression.**



We let F_s be the **resultant** compressive force in the sleeve, the sum of the distributed loading around the circumference. F_b is the tensile force in the bolt. Like the carton-tie-down exercise, these two internal forces are *self equilibrating*; there are no external applied forces in the final state. We have $F_b - F_s = 0$

The normal stresses in the sleeve and the bolt are found assuming the resultant forces of tension and compression are uniformly distributed over their respective areas. Equilibrium then can be expressed as $\sigma_b \cdot A_b = \sigma_s \cdot A_s$ where the *A*'s are the cross sectional areas of the bolt and of the sleeve.

Constitutive Relations

The constitutive relations are, for uniaxial loading, which is the case we have on hand,

$$\sigma_s = E_s \cdot (d_s / L) \qquad \text{and} \qquad \sigma_b = E_b \cdot (d_b / L)$$

1. Since this is a *thought experiment* we don't have to worry about the details of this physically impossible move.

We have then a total of four equations for four unknowns – the two displacements, the two stresses. Substituting the expressions for the stresses in terms of displacements into the equilibrium allows me to write $d_b = d_s \cdot (A_s E_s / A_b E_b)$

which tells me the relative deformation as a function of the relative stiffness of the two material. If the sleeve is “softer”, the bolt deforms less... etc.

With this, compatibility gives me a way to solve for the displacements in terms of Δ. I obtain, letting

$$\beta = (A_s E_s / A_b E_b)$$

we have

$$d_s = \Delta \frac{1}{(1+\beta)} \qquad \text{and} \qquad d_b = \Delta \frac{\beta}{(1+\beta)}$$

Values for the stresses are found to be $\sigma_b = 79\ MN/m^2$ *and* $\sigma_s = 63\ MN/m^2$. (Note: compressive)

In computing these values, the elastic modulus for steel and aluminum were taken as 200 GN/m^2 and 70 GN/m^2 respectively. Observe that, though the steel experiences less strain, its stress level is greater in magnitude than that seen in the aluminum.

4.10 Modes of Failure

The *failure* of structures occurs in a variety of ways. We have talked about *yielding, the onset of plastic flow* of *ductile* materials, materials which show relatively large, even sensible, deformations for relatively small increases in load once the material is loaded beyond its *yield strength*. If the excessive load is removed before complete collapse, the structure will not wholly return to its original, undeformed configuration.

Although it is the tension test that is used to fix the limit of elastic behavior and to define a *yield strength*, the mechanism for yielding is a shearing action of the material on a microscopic level. We have seen how a tensile stress in a bar can produce a shear stress on a plane inclined to the axis of the bar. We will pursue this phenomenon further in a subsequent chapter. There is nothing here that contradicts our symmetry arguments which ruled out the possibility of a shear strain being generated by a normal stress. Our symmetry arguments only applied to components of stress and strain relative to a single orientation of our reference axis at a point.

Not all materials behave as steel or aluminum or ductile plastics. Some materials are *brittle*. Load glass, chalk, cast iron, a brittle plastic, a carbon fiber, or concrete in a tension test and they will break with very little extension. They show insensible deformation all the way up to the fracture load. Material properties and modes of failure also may depend upon temperature. What may be ductile at room temperature will be brittle at low temperatures. In compression, a brittle material can carry a significantly greater load before fracture. The collapse or fracture of a structure are not the only modes of failure. At high temperatures, still well below the melting point, materials will *creep* – they will continuously deform at a constant load. A material can fail in *fatigue*: Under continual cycling through tension then compression, a material will fail well below the yield strength or fracture stress witnessed in a tension test. Some failure modes involve a more *macroscopic* behavior of the material. *Delamination* of a multilayered, glued together material is one such mode. Another of particular interest in this course is *elastic buckling*, a mode in which large deflections and collapse of a structure may occur well before any limiting load defined by a tension test are reached. Excessive deflections themselves, even though small relative to those which might be seen in elastic buckling or plastic collapse of a structure, can be considered “failure”, at least a failure of design. Think of the constraints on deflections and rotations that must be satisfied by the structural support of an optical telescopic or another instrument in which alignment is critical.

Failure, then has many faces. Most if not all of its modes depend upon knowing well a structure’s behavior in terms of the variables we have constructed here: stress, strain, and displacements.

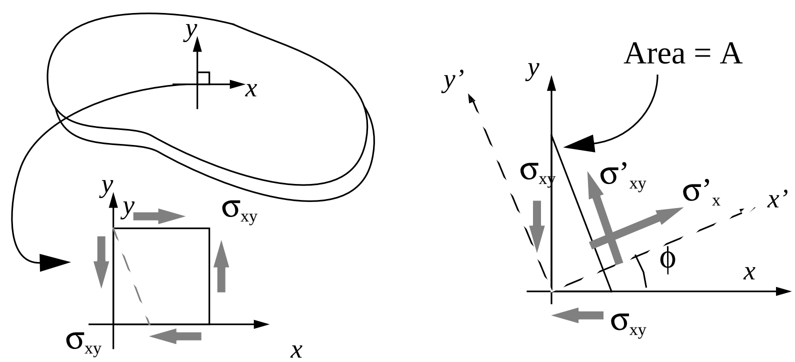
4.11 Problems

4.1 A fluid can be defined as a continuum which is unable to support a shear stress and remain at rest — like a rigid body. The state of stress at any point, within a fluid column for example, we label “hydrostatic”; the normal stress components are equal to the negative of the static pressure at the point and the shear stress components are all zero. $\sigma_x = \sigma_y = \sigma_z = -p$ and $\sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$.

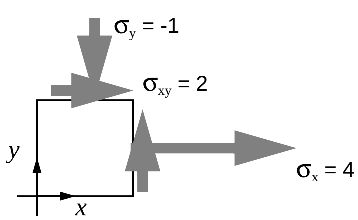
Using the two dimensional transformation relations (the existence of σ_z does not affect their validity) show that the shear stress on any arbitrarily oriented plane is zero and the normal stress is again -p.

4.2 Estimate the compressive stress at the base of the Washington Monument - the one on the Maill in Washington, DC.

4.3 The stress at a point in the plane of a thin plate is shown. Only the shear stress component is not zero relative to the x-y axis. From equilibrium of a section cut at the angle ϕ**,** deduce expressions for the normal and shear stress components acting on the inclined face of area A. NB: stress is a force per unit area so the areas of the faces the stress components act upon must enter into your equilibrium considerations.



4.4 Construct Mohr’s circle for the state of stress of exercise **4.3**, above. Determine the "principle stresses"and the orientation of the planes upon which they act relative to the *xy* frame.



4.5 Given the components of stress relative to an x-y frame at a point in plane stress are:

$\sigma_x = 4$, $\quad \sigma_{xy} = 2$ $\quad \sigma_y = -1$

What are the components with respect to an axis system rotated 30 deg. counter clockwise at the point?

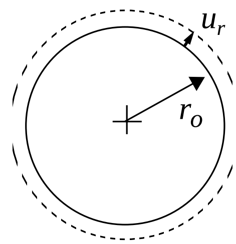
Determine the orientation of axis which yields maximum and minimum normal stress components. What are their values?

4.6 A thin walled glass tube of radius R = 1 inch, and wall thickness t= 0.010 inches, is closed at both ends and contains a fluid under pressure, p = 100 psi. A torque, M_t, of 300 inch-lbs, is applied about the axis of the tube.

Compute the stress components relative to a coordinate frame with its x axis in the direction of the tube’s axis, its y axis circumferentially directed and tangent to the surface.

Determine the maximum tensile stress and the orientation of the plane upon which it acts.

4.7 *Show that* for the thin circular hoop subject to an axi-symmetric, radial extension u_r, that the circumferential extensional strain, can be expressed as

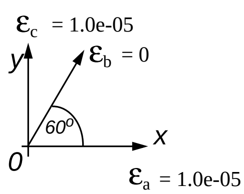


$$\varepsilon_\theta = (L - L_o)/L_o = u_r/r_o$$

where L_o is the original, undeformed circumference.

4.8 *What if* we change our sign convention on stress components so that a normal, compressive stress is taken as a positive quantity (a tensile stress would then be negative). What becomes of the transformation relations? How would you alter the rules for constructing and using a Mohr's circle to find the stress components on an arbitrarily oriented plane?

What if you changed your sign convention on shear stress as well; how would things change?



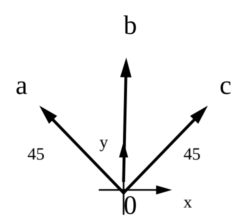
4.9 Three strain gages are mounted in the directions shown on the surface of a thin plate. The values of the extensional strain each measures is also shown in the figure.

i) Determine the shear strain component γ_{xy} at the point with respect to the *xy* axes shown.

ii) What orientation of axes gives extreme values for the extensional strain components at the point.

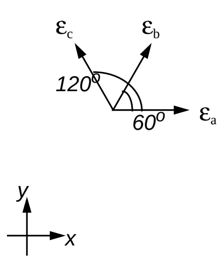
iii) What are these values.

4.10 Three strain gages measure the extensional strain in the three directions 0a, 0b and 0c at "the point 0". Using the relationship we derived in class



$$\varepsilon_{PQ} = \varepsilon_x \cos^2\phi + \gamma_{xy} \cos\phi \sin\phi + \varepsilon_y \sin^2\phi$$

find the components of strain with respect to the xy axis in terms of ε_a, ε_b and ε_c



4.11 A strain gage rosette, fixed to a flat, thin plate, measures the following extensional strains

ε_a = 1. E-04

ε_b = 1. E-04

ε_c = 2. E-04

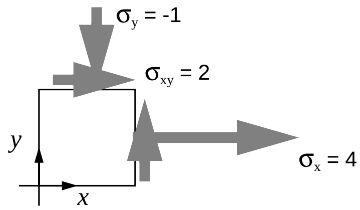
Determine the state of strain at the point, expressed in terms of components relative to the xy coordinate frame shown.

4.12 Given the components of stress relative to an x-y frame at a point in plane stress are:

$\sigma_x = 4, \quad \sigma_{xy} = 2 \quad \sigma_y = -1$

What are the components with respect to an axis system rotated 30 deg. counter clockwise at the point?

Determine the orientation of axis which yields maximum and minimum normal stress components. What are their values?



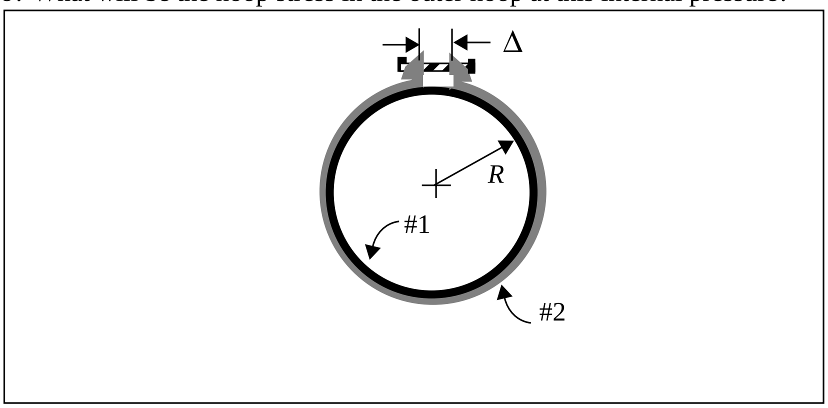
4.13 Hoop #1 is enclosed within hoop #2. The two are made of different materials, have different thicknesses but the same width (into the page). They are shown in their unstressed state, just touching. *Show that* after tightening the bolt at the top of the assembly and closing the gap, Δ, to zero, the stress in the outer hoop is tensile and has magnitude $F/(bt_1)$ while the stress in the inner hoop is compressive and has magnitude $F/(bt_2)$. In these t_1 and t_2 are the thicknesses,

$$F = k_1 k_2 \Delta/(k_1+k_2)$$

where

$$k_1 = (bt_1)E_1/L_1 \quad \text{and} \quad k_2 = (bt_2)E_2/L_2$$

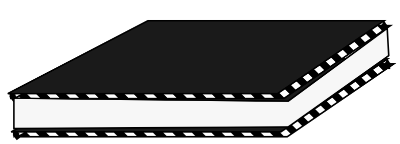
What if an internal pressure is applied to the inner hoop? When will the stress in the inner hoop diminish to zero? What will be the hoop stress in the outer hoop at this internal pressure?



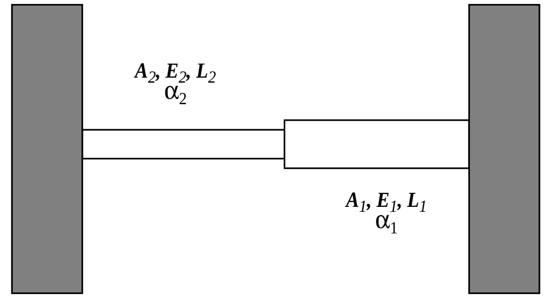
4.14 The thin plate is a composite of two materials. A quarter inch thick, steel, plate is clad on both sides with a thin (t_{al} = 0.005 in), uniform, layer of aluminum. The structure is stress-free at room temperature. *Show that* the stresses generated in the two materials, when the temperature changes an amount ΔT, may be approximated by

$$\sigma_{al} = (\alpha_{st} - \alpha_{al})E_{al}\,\Delta T/(1-\nu) \quad \text{and} \quad \sigma_{st} = -(2t_{al}/t_{st})(\alpha_{st}-\alpha_{al})E_{al}\Delta T/(1-\nu)$$

At what temperature will the clad plate begin to plastically deform? Where?



4.15 Two cylindrical rods, of two different materials are rigidly restrained at the ends where they meet the side walls. The system is subject to a temperature increase ΔT



How must their properties be related if the point at which they meet is not to move left or right?

If material #1 is steel and #2 is aluminum, what more specifically can you say?

E_1 = 200 GPa steel

E_2 = 70 GPa aluminum α_1 = 15 e-06 $/^oC$ α_2 = 23 e-06 $/^oC$

Design Exercise 4.1

A solid circular steel shaft of diameter 40 *mm* is to be fitted with a thin-walled circular cylindrical sleeve, *also made of steel*. In service the system is to serve as a stop, halting the motion of another fitted, but freely moving cylindrical tube whose inner radius is slightly larger than the outer radius of the solid shaft. The stopping sleeve is to remain in fixed location on the solid shaft for all axial loads less than some critical value of the force *F* shown in the figure. That is, for $F < 50kN$. If *F* exceeds this limit the sleeve is to frictionally break free and allow the sliding cylinder to continue moving. along the shaft.

It is proposed to fasten the sleeve to the shaft by means of a *shrink fit*. The initial inner radius of the sleeve is to be made slightly *less* than the initial outer radius of the shaft. The sleeve is then heated to a temperature not to exceed $\Delta T_{max} = 250^oC$ so that its heat-treatment is not affected. The *hot* sleeve is then slipped over the shaft and positioned as desired. When the sleeve cools down, the radial *misfit* between the shaft outer radius and the sleeve's unstressed inner radius will generate sufficient mechanical interaction between the two so that the stopping and break-away functions can be fulfilled.

Size the sleeve.