

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

8.02

Spring 2013

Exam 3 Equation Sheet

Force Law: $\vec{F}_q = q(\vec{E}_{ext} + \vec{v}_q \times \vec{B}_{ext})$

Force on Current Carrying Wire:

$$\vec{F} = \int_{\text{wire}} Id\vec{s}' \times \vec{B}_{ext}$$

Magnetic Dipole

$$\vec{\mu} = IA\hat{n}_{RHR}$$

Torque on a Magnetic Dipole

$$\vec{\tau} = \vec{\mu} \times \vec{B}_{ext}$$

Force on a Magnetic Dipole

$$\vec{F} = \vec{\nabla}(\vec{\mu} \cdot \vec{B})$$

$$F_z = \mu_z \frac{\partial B_z}{\partial z}$$

Source Equations:

$$\vec{E}(\vec{r}) = k_e \int_{\text{source}} \frac{dq}{r^2} \hat{r} = k_e \int_{\text{source}} \frac{dq'(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

$$\vec{B}(\vec{r}) = \frac{\mu_o}{4\pi} \int_{\text{source}} \frac{Id\vec{s}' \times \hat{r}}{r^2}$$

$$\vec{B}(\vec{r}) = \frac{\mu_o}{4\pi} \int_{\text{source}} \frac{Id\vec{s}' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

\hat{r} points from source to field point

Ampere's Law

$$\oint_{\text{closed path}} \vec{B} \cdot d\vec{s} = \mu_o \iint_S \vec{J} \cdot \hat{n} da$$

Faraday's law

$$\oint_{\text{closed fixed path}} \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \iint_{\text{open surface } S} \vec{B} \cdot \hat{n} da$$

$$\mathcal{E} = I_{ind} R$$

Gauss's Law for Magnetism:

$$\oiint_{\text{closed surface}} \vec{B} \cdot \hat{n} da = 0$$

Gauss's Law:

$$\oiint_{\text{closed surface}} \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_o}$$

Electric Potential Difference

Electrostatics:

$$\Delta V = V_b - V_a \equiv -\int_a^b \vec{E} \cdot d\vec{s}$$

$$\vec{E} = -\vec{\nabla}V$$

Potential Energy:

$$\Delta U = q\Delta V$$

Capacitance:

$$C = \frac{Q}{\Delta V} \quad U_E = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} C \Delta V^2$$

Inductance: $L = \frac{\Phi_{B,Total}}{I} = \frac{N\Phi_B}{I}$

$$\mathcal{E}_{back} = -LdI/dt \quad U_M = \frac{1}{2} LI^2$$

Energy Density Stored in Fields:

$$u_E = \frac{1}{2} \epsilon_o E^2 \quad ; \quad u_B = \frac{1}{2} B^2 / \mu_o$$

Current Density and Current:

$$I = \iint_{\text{open surface}} \vec{J} \cdot \hat{n} \, da$$

Ohm's Law: $\Delta V = I R$

$$\vec{J} = \sigma_c \vec{E} \text{ where } \sigma_c \text{ is the conductivity}$$

$$\vec{E} = \rho_r \vec{J} \text{ where } \rho_r \text{ is the resistivity}$$

Power: $P = \vec{F} \cdot \vec{v}$

Power from Voltage Source:

$$P_{\text{source}} = I \Delta V$$

Power Dissipated in Resistor:

$$P_{\text{Joule}} = I^2 R = \Delta V^2 / R$$

Constants:

$$\mu_0 \equiv 4\pi \times 10^{-7} \text{ T} \cdot \text{m} \cdot \text{A}^{-1}$$

$$\epsilon_0 \equiv 1 / \mu_0 c^2 \simeq 8.85 \times 10^{-12} \text{ C}^2 \cdot \text{N}^{-1} \cdot \text{m}^{-2}$$

$$k_e \equiv 1 / 4\pi\epsilon_0 \simeq 9.0 \times 10^9 \text{ N} \cdot \text{m}^2 \cdot \text{C}^{-2}$$

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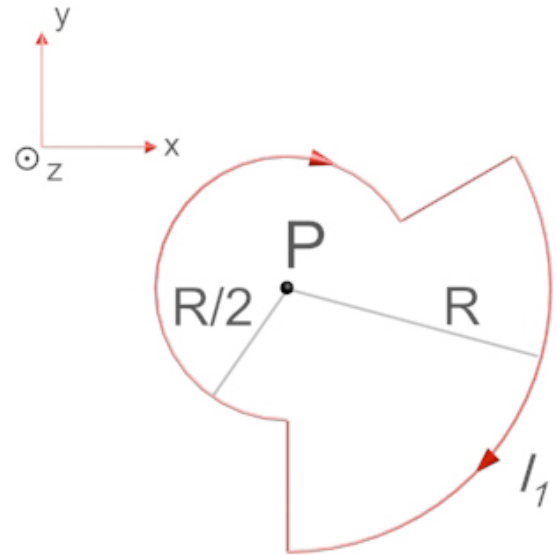
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Exam 3 Practice Problems Solutions

Problem 1 Biot Savart

A current loop, shown in the figure below, consists of two arc segments as shown, with a common center at P . One arc segment has an opening angle of 120 degrees and the other arc segment has an opening angle of 240 degrees. Two straight line segments join the arc segments. One arc segment has radius R and the other arc segment has radius $R/2$. A current I_1 flows clockwise in the loop in the direction shown.



- a) What is the direction and magnitude of the magnetic field \vec{B} at the point P ?

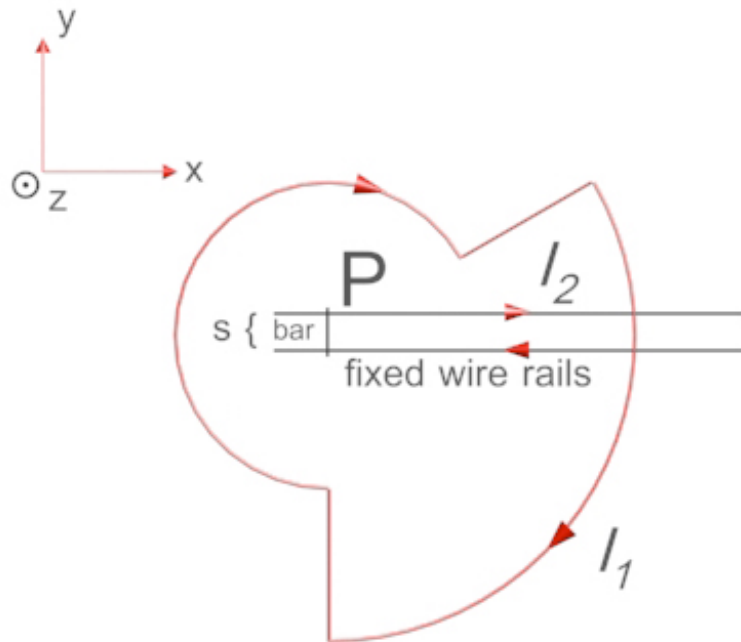
Answer: The sides will not contribute because the cross product is zero there.

$$\vec{B}_{\text{outer arc}}(\vec{r}) = \frac{\mu_o}{4\pi} \int \frac{I_1 d\vec{s} \times \hat{r}}{r^2} = -\hat{z} \frac{\mu_o I_1}{4\pi R^2} \int ds = -\hat{z} \frac{\mu_o I_1}{4\pi R^2} \frac{1}{3} (2\pi R) = -\hat{z} \frac{\mu_o I_1}{6R}$$

$$\vec{B}_{\text{inner arc}}(\vec{r}) = \frac{\mu_o}{4\pi} \int \frac{I_1 d\vec{s} \times \hat{r}}{r^2} = -\hat{z} \frac{\mu_o I_1}{4\pi (R/2)^2} \int ds = -\hat{z} \frac{\mu_o I_1}{4\pi (R/2)^2} \frac{2}{3} 2\pi \frac{R}{2} = -\hat{z} \frac{2\mu_o I_1}{3R}$$

Therefore the total field at P is given by $\vec{B}(\vec{r}) = -\hat{z} \frac{\mu_o I_1}{6R} - \hat{z} \frac{4\mu_o I_1}{6R} = -\hat{z} \frac{5\mu_o I_1}{6R}$

Two fixed conducting rails are arranged as shown in the figure below. A metal bar of length s is placed at the origin, initially held in place, and a current I_2 runs through the rails and bar clockwise. The bar is then released. *You may assume that the length of the bar is very short and that the magnetic field you calculated in part a) is uniform over the length of the bar. You may neglect the magnetic field due to the current through the rails.*



- b) What is the direction and magnitude of the magnetic force acting on bar the instant it is released?

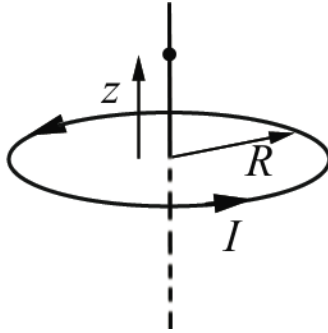
Answer: The field at point P is inward from our calculations above, and the current in the bar is upward, so the total force on the bar is in the negative x direction, with

c)

$$\vec{F} = -\hat{x} \frac{5\mu_0 I_1}{6R} I_2 s$$

Problem 2: Current Ring

A circular ring of radius R has a current I in the counterclockwise direction as seen from above.



- c) Calculate the magnetic field due to the current at an arbitrary point a distance z along the z -axis passing through the center of the ring perpendicular to the plane of the ring.

- d) Calculate the line integral $\int_{z=-\infty}^{z=+\infty} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ along the z -axis from $z = -\infty$ to $z = +\infty$.

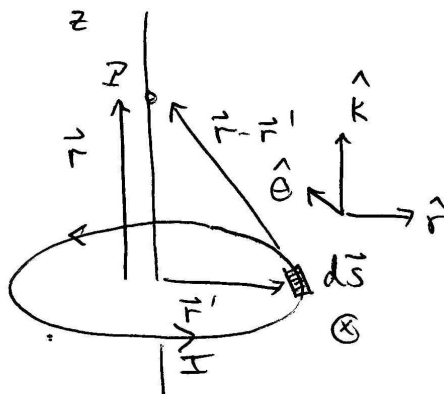
- e) The path is not a closed and yet it satisfies Ampere's Law. Why?

Solution:

We shall apply the Biot-Savart law to calculate the magnetic field at the point P a distance z along the positive z -axis

$$\vec{\mathbf{B}}(P) = \int_{\theta=0}^{\theta=2\pi} \frac{\mu_0 I}{4\pi} \frac{d\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2} = \int_{\theta=0}^{\theta=2\pi} \frac{\mu_0 I}{4\pi} \frac{d\vec{\mathbf{s}} \times (\vec{\mathbf{r}} - \vec{\mathbf{r}}')}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^3}$$

Choose a coordinate system and unit vectors as shown in the figure below.



Then $d\vec{s} = R d\theta \hat{\theta}$, $\vec{r} = z \hat{k}$, and $\vec{r}' = R \hat{r}$. Then the Biot Savart Law becomes

$$\vec{B}(z) = \int_{\theta=0}^{\theta=2\pi} \frac{\mu_0 I}{4\pi} \frac{(R d\theta \hat{\theta}) \times (z \hat{k} - R \hat{r})}{|z \hat{k} - R \hat{r}|^3}. \quad (1.1)$$

We now explicitly calculate the cross products noting that $\hat{\theta} \times \hat{k} = \hat{r}$ and $\hat{\theta} \times (-\hat{r}) = \hat{k}$ and Eq. (1.1) becomes

$$\vec{B}(z) = \frac{\mu_0 I z R}{4\pi(z^2 + R^2)^{3/2}} \int_{\theta=0}^{\theta=2\pi} d\theta \hat{r} + \frac{\mu_0 I R^2}{4\pi(z^2 + R^2)^{3/2}} \int_{\theta=0}^{\theta=2\pi} d\theta \hat{k} \quad (1.2)$$

The first integral vanishes because as we integrate around the circle the unit vector \hat{r} always points radially outward and hence sums to zero. Equivalently if we choose coordinates such that $\hat{r} = \cos\theta \hat{i} + \sin\theta \hat{j}$ then

$$\begin{aligned} \vec{B}(z) &= \frac{\mu_0 I z R}{4\pi(z^2 + R^2)^{3/2}} \int_{\theta=0}^{\theta=2\pi} d\theta \hat{r} = \frac{\mu_0 I z R}{4\pi(z^2 + R^2)^{3/2}} \int_{\theta=0}^{\theta=2\pi} d\theta (\cos\theta \hat{i} + \sin\theta \hat{j}) \\ &= \frac{\mu_0 I z R}{4\pi(z^2 + R^2)^{3/2}} (\sin\theta \Big|_{\theta=0}^{\theta=2\pi} \hat{i} + -\cos\theta \Big|_{\theta=0}^{\theta=2\pi} \hat{j}) = \vec{0} \end{aligned}$$

The second integral in Eq. (1.2) is straightforward and so the magnetic field along the z-axis is given by

$$\vec{B}(z) = \frac{\mu_0 I R^2}{4\pi(z^2 + R^2)^{3/2}} \theta \Big|_{\theta=0}^{\theta=2\pi} \hat{k} = \frac{\mu_0 I R^2}{2(z^2 + R^2)^{3/2}} \hat{k}. \quad (1.3)$$

We now calculate the line integral $\int_{z=-\infty}^{z=+\infty} \vec{B} \cdot d\vec{s}$ along the z-axis from $z = -\infty$ to $z = +\infty$.

$$\int_{z=-\infty}^{z=+\infty} \vec{B} \cdot d\vec{s} = \int_{z=-\infty}^{z=+\infty} \frac{\mu_0 I R^2}{2(z^2 + R^2)^{3/2}} \hat{k} \cdot dz \hat{k} = \int_{z=-\infty}^{z=+\infty} \frac{\mu_0 I R^2}{2(z^2 + R^2)^{3/2}} dz \quad (1.4)$$

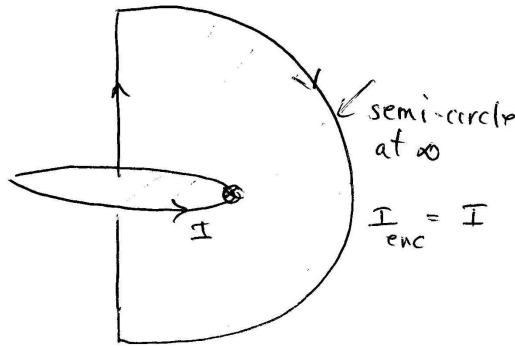
This integral is equal to

$$\int_{z=-\infty}^{z=+\infty} \vec{B} \cdot d\vec{s} = \int_{z=-\infty}^{z=+\infty} \frac{\mu_0 I R^2}{2(z^2 + R^2)^{3/2}} dz = \frac{\mu_0 I z}{2(z^2 + R^2)^{1/2}} \Big|_{z=-\infty}^{z=+\infty} = \frac{\mu_0 I}{2} - -\frac{\mu_0 I}{2} = \mu_0 I \quad (1.5)$$

where we have used the fact that

$$\lim_{z \rightarrow \infty} \frac{\mu_0 I(z)}{2((z)^2 + R^2)^{1/2}} = \frac{\mu_0 I}{2} \quad \text{and} \quad \lim_{z \rightarrow -\infty} \frac{\mu_0 I(z)}{2((z)^2 + R^2)^{1/2}} = -\frac{\mu_0 I}{2}.$$

Recall that Ampere's Law states that $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{enc}$ however the integral above is not a closed line integral.



However if we integrate $\vec{\mathbf{B}}$ along a semi-circular path at infinity from $z = +\infty$ to $z = -\infty$ and add our two line integrals together we get a closed line integral

$$\int_{z=-\infty}^{z=+\infty} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} + \int_{\text{semi-circle at infinity}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} \quad (1.6)$$

Although we have only calculated $\vec{\mathbf{B}}(z)$ along the z -axis, we see that

$$\lim_{z \rightarrow \infty} \vec{\mathbf{B}}(z) = \lim_{z \rightarrow \infty} \frac{\mu_0 I R^2}{2(z^2 + R^2)^{3/2}} \hat{\mathbf{k}} = \vec{\mathbf{0}}. \quad \text{This holds everywhere on the semi-circle at infinity}$$

so

$$\int_{\text{semi-circle at infinity}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \vec{\mathbf{0}} \quad (1.7)$$

Therefore combining Eqs. (1.7), (1.6), and (1.5) we have that

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I \quad (1.8)$$

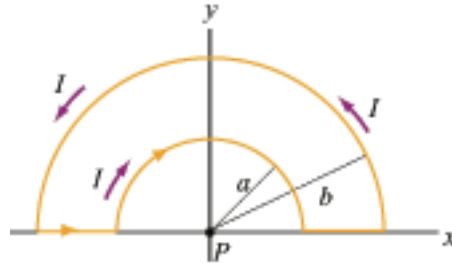
From the figure above we see that

$$I_{enc} = I \quad (1.9)$$

Therefore Ampere's Law is satisfied.

Problem 3: Magnetic fields of Current Loops

(a) Find the magnetic field at point P due to current loop shown in the figure below.



Solution:

(b) There is no magnetic field due to the straight segments because point P is along the lines. Using the general expression for $d\vec{B}$ obtained in (a), for the outer segment, we have

$$\vec{B}_{\text{out}} = \int_0^{\pi} \frac{\mu_0}{4\pi} \frac{Id\theta}{b} \hat{k} = \left(\frac{\mu_0 I}{4b} \right) \hat{k}$$

Similarly, the contribution to the magnetic field from the inner segment is

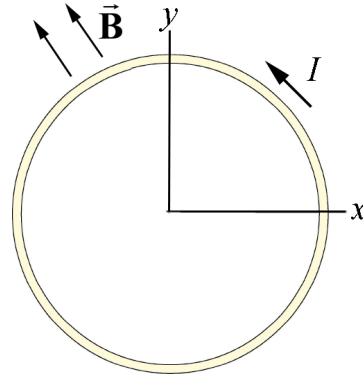
$$\vec{B}_{\text{in}} = \int_{\pi}^0 \frac{\mu_0}{4\pi} \frac{Id\theta}{a} \hat{k} = - \left(\frac{\mu_0 I}{4a} \right) \hat{k}$$

Therefore the net magnetic field at Point P is

$$\vec{B}_{\text{net}} = \vec{B}_{\text{out}} + \vec{B}_{\text{in}} = - \frac{\mu_0 I}{4} \left(\frac{1}{a} - \frac{1}{b} \right) \hat{k} .$$

Problem 4: Torque on Circular Current Loop

A wire ring lying in the xy -plane with its center at the origin carries a counterclockwise current I . There is an external uniform magnetic field $\vec{\mathbf{B}} = B_x \hat{\mathbf{i}} + B_y \hat{\mathbf{j}}$ such that $B_y > 0$ and $B_x < 0$. The magnetic moment vector $\vec{\boldsymbol{\mu}}$ is perpendicular to the plane of the loop and has magnitude $\mu = IA$ and the direction is given by right-hand-rule with respect to the direction of the current. What is the direction and magnitude of the torque on the loop?



Solution: The torque on a current loop in a uniform field is given by

$$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\mathbf{B}},$$

where $\vec{\boldsymbol{\mu}}$ is the magnetic dipole moment with magnitude $|\vec{\boldsymbol{\mu}}| = IA$ and points perpendicular to the plane of the loop and right-handed with respect to the direction of current flow. The magnetic dipole moment is given by

$$\vec{\boldsymbol{\mu}} = I\vec{\mathbf{A}} = I\pi R^2 \hat{\mathbf{k}}.$$

Therefore,

$$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\mathbf{B}} = (I\pi R^2) \hat{\mathbf{k}} \times (B_x \hat{\mathbf{i}} + B_y \hat{\mathbf{j}}) = I\pi R^2 (B_x \hat{\mathbf{j}} - B_y \hat{\mathbf{i}}).$$

Define an angle θ between the negative x - and y -axes as shown in the figure to the right. Then the direction of the torque is given by

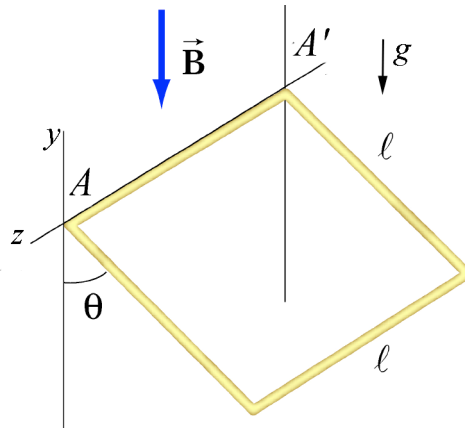
$$\theta = \tan(-B_x / B_y) > 0$$

and the magnitude is

$$|\vec{\boldsymbol{\tau}}| = I\pi R^2 \sqrt{B_x^2 + B_y^2}.$$

Problem 5: Torque and Force

A square loop of wire, of length ℓ on each side, and mass m , pivots about an axis AA' that corresponds to a horizontal side of the square, as shown in the figure on the left below. The external magnetic field $\vec{\mathbf{B}}$ of magnitude B is directed vertically downward, and uniformly fills the region in the vicinity of the loop. A current I flows around the loop. The gravitational torque on the loop and the magnetic torque on the loop sum to zero when the loop makes an angle θ with the z -axis. The magnitude of the gravitational field is $g = 9.8 \text{ m} \cdot \text{s}^{-2}$.

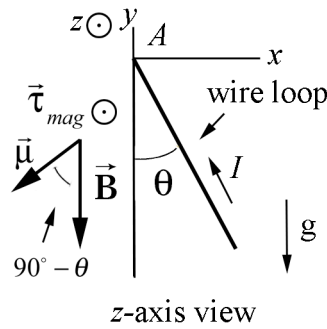


- In what direction does the current need to flow in order that the magnetic torque acts in an opposite direction from the gravitational torque?
- Calculate the magnitude of the magnetic torque on this loop of wire in terms of the quantities given.
- Suppose that the mass of the loop $m = 0.4 \text{ kg}$ and the length of a side is $l = 1.0 \text{ m}$. Suppose that when current in the loop is $I = 2.0 \text{ A}$, the torques on the loop balance when $\theta = 45^\circ$. What is the magnitude of the magnetic field?

Solutions:

- In what direction does the current need to flow in order that the magnetic torque acts in an opposite direction from the gravitational torque?

Answer: Because the gravitational torque acts in the negative z -direction, the magnetic torque must act in the positive z -direction. The magnetic torque is given by the expression $\vec{\tau}_{mag} = \vec{\mu} \times \vec{\mathbf{B}}$. The magnetic field $\vec{\mathbf{B}} = -B\hat{\mathbf{j}}$ and so the magnetic dipole moment $\vec{\mu} = \mu_x \hat{\mathbf{i}} + \mu_y \hat{\mathbf{j}}$ must point towards the lower left (third quadrant) with both negative x - and y -components, i.e. $\mu_x < 0$, and $\mu_y < 0$. Therefore the current must flow in the clockwise direction as seen from above.



- b) Calculate the magnitude of the magnetic torque on this loop of wire in terms of the quantities given.

Answer: We know that $\vec{\tau}_{mag} = \vec{\mu} \times \vec{B}$ and that the magnitude of the magnetic moment of the loop is $\mu = IA = I\ell^2$. Note that the angle between the magnetic field and the magnetic moment is $90^\circ - \theta$, therefore the magnitude of the magnetic torque is

$$\tau_{mag} = I\ell^2 B \sin(90^\circ - \theta) = I\ell^2 B \cos(\theta)$$

- c) Suppose that the mass of the loop $m = 0.4 \text{ kg}$ and the length of a side is $l = 1.0 \text{ m}$. Suppose that when current in the loop is $I = 2.0 \text{ A}$, the torques on the loop balance when $\theta = 45^\circ$. What is the magnitude of the magnetic field?

Answer: Because the loop is pivoted, we need to consider the gravitational torque on bottom leg and the two side legs. The gravitational torque is

$$\vec{\tau}_{grav} = (m/4)g\ell \sin(\theta)(-\hat{k}) + 2(m/4)g(\ell/2) \sin(\theta)(-\hat{k}) = (m/2)g\ell \sin(\theta)(-\hat{k}).$$

The magnetic torque is $\vec{\tau}_{mag} = I\ell^2 B \cos(\theta)(\hat{k})$ and so the torques balance when

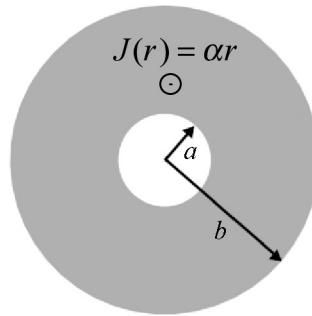
$$I\ell^2 B \cos(\theta) = (m/2)g\ell \sin(\theta).$$

Therefore the magnitude of the magnetic field is

$$B = \frac{mg\ell \sin(\theta)}{2I\ell^2 \cos(\theta)} = \frac{mg \tan(\theta)}{2I\ell} = \frac{(0.4 \text{ kg})(9.8 \text{ m s}^{-2})(1)}{2(2.0 \text{ A})(1 \text{ m})} = 1.0 \text{ T}.$$

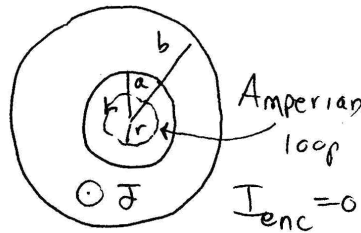
Problem 6: Ampere's Law

Non-Uniform Current Density A long cylindrical cable consists of a conducting cylindrical shell of inner radius a and outer radius b . The current density \vec{J} in the shell is out of the page (see sketch) and varies with radius as $J(r) = \alpha r$ for $a < r < b$ and is zero outside of that range, where α is a positive constant with units $A \cdot m^{-3}$. Find the magnetic field in each of the following regions, indicating both magnitude and direction (i) $r < a$, (ii) $a < r < b$, and (iii) $r > b$. For each region make a figure clearly showing your choice of Amperian loop.



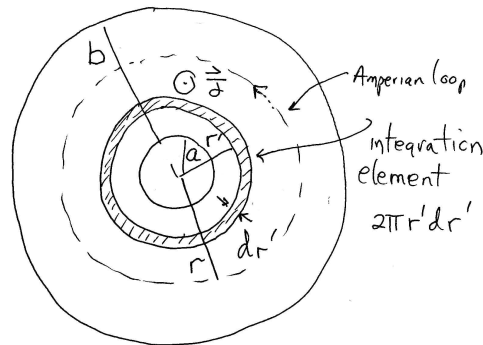
Solution:

(a) For $r < a$, the Amperian loop is shown in the figure below.



The enclosed current is $I_{enc} = 0$. Therefore, by Ampere's law, $B = 0$

(b) In the region $a < r < b$, the Amperian loop and integration element is shown in the figure below.



The enclosed current is

$$I_{\text{enc}} = \iint_{\text{open surface}} \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}} = \int_a^r \alpha r' 2\pi r' dr' = 2\pi\alpha \int_a^r r'^2 dr' = \frac{2\pi\alpha}{3}(r^3 - a^3)$$

Applying Ampere's law, we have

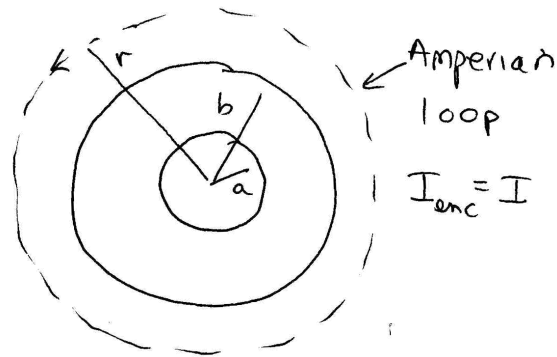
$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{\text{enc}} \Rightarrow B(2\pi r) = \mu_0 \frac{2\pi\alpha}{3}(r^3 - a^3)$$

or

$$\vec{\mathbf{B}} = B\hat{\phi} = \mu_0 \frac{\alpha}{3}(r^3 - a^3) \frac{1}{r} \hat{\phi}$$

The magnetic field points in the azimuthal direction.

(b) In the region $r > b$, the Amperian loop is shown in the figure below. We use the same integration element as in part b).



Then the enclosed current is

$$I_{\text{enc}} = \iint_{\text{open surface}} \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}} = \int_a^b \alpha r' 2\pi r' dr' = \frac{2\pi\alpha}{3} \int_a^b r'^2 dr' = \frac{2\pi\alpha}{3}(b^3 - a^3)$$

Applying Ampere's law, we have

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{\text{enc}} \Rightarrow B(2\pi r) = \frac{\mu_0 2\pi\alpha}{3}(b^3 - a^3)$$

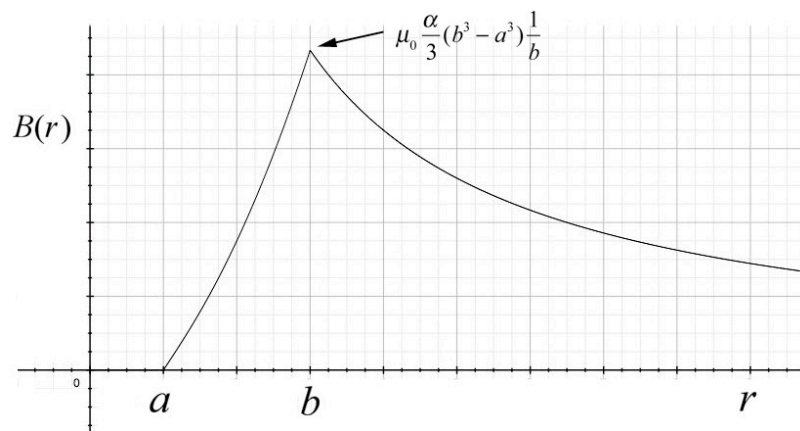
or

$$\vec{\mathbf{B}} = B\hat{\phi} = \mu_0 \frac{\alpha}{3}(b^3 - a^3) \frac{1}{r} \hat{\phi}$$

Collecting our results we have that

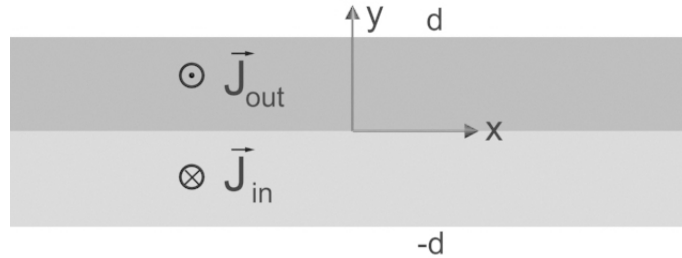
$$\vec{\mathbf{B}} = \begin{cases} \vec{\mathbf{0}}; & r < a \\ \mu_0 \frac{\alpha}{3} (r^3 - a^3) \frac{1}{r} \hat{\phi}; & a < r < b \\ \mu_0 \frac{\alpha}{3} (b^3 - a^3) \frac{1}{r} \hat{\phi} ; & r > b \end{cases}$$

A graph of the magnitude of B vs. r is shown below.



Problem 7: Ampere's Law

The figure below shows two slabs of current. Both slabs of current are infinite in the x and z directions, and have thickness d in the y -direction. The top slab of current is located in the region $0 < y < d$ and has a constant current density $\vec{J}_{out} = J\hat{z}$ out of the page. The bottom slab of current is located in the region $-d < y < 0$ and has a constant current density $\vec{J}_{in} = -J\hat{z}$ into the page.



(a) What is the magnetic field for $|y| > d$? Justify your answer.

The magnitude of the magnetic field of a single slab of current of thickness d outside the slab is $\mu_0 Jd / 2$, and the field reverses depending on the sign of y , with $|y| > d$. The field of the two slabs shown above is zero for $|y| > d$ because when you are above the sheet or below the sheet by superposition of the field due to the slab carrying current out of the page just cancels the field due to the slab carrying current into the page.

(b) Use Ampere's Law to find the magnetic field at $y = 0$. Show the Amperian Loop that you use and give the magnitude and direction of the magnetic field.

We take an Amperian loop whose bottom is at $y = 0$ and whose top is at $y \geq d$, of width w . We have

$$\oint_{\text{closed path}} \vec{B} \cdot d\vec{s} = w B_x \Big|_{y=0} = \mu_0 J w d \Rightarrow \vec{B} \Big|_{y=0} = \hat{x} \mu_0 J d$$

(c) Use Ampere's Law to find the magnetic field for $0 < y < d$. Show the Amperian Loop that you use and give the magnitude and direction of the magnetic field.

We take an Amperian loop whose bottom is at $y = 0$ and whose top is at $0 \leq y \leq d$, of width w . We have

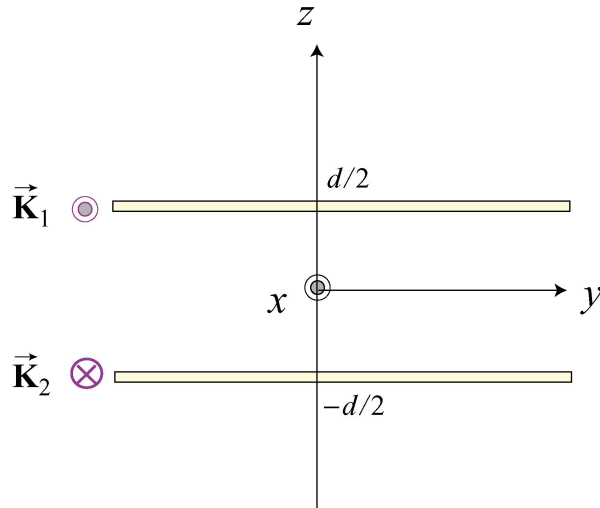
$$\oint_{\text{closed path}} \vec{B} \cdot d\vec{s} = w B_x \Big|_{y=0} - w B_x \Big|_{y=y} = \mu_0 J w d - w B_x \Big|_{y=y} = \mu_0 J w y \Rightarrow \vec{B} \Big|_{y=y} = \hat{x} \mu_0 J (d - y)$$

(d) Make a plot of the x -component of the magnetic field as a function of the distance y .

The plot starts out at $\mu_0 J d$ at $y=0$ and then falls to zero linearly, both going upward and downward, going to zero at $|y|=d$ and staying zero thereafter.

Problem 8: Two Current Sheets Ampere's Law

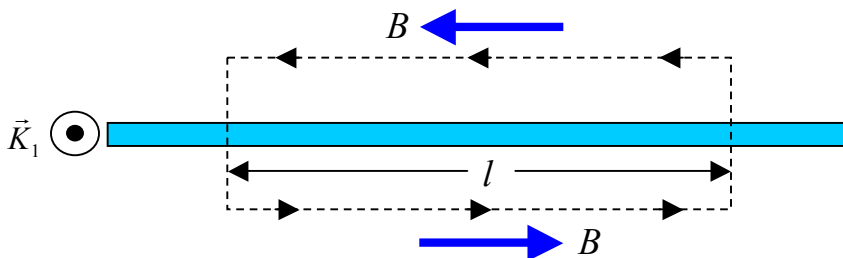
Consider two infinitely large sheets lying in the xy -plane separated by a distance d carrying surface current densities $\vec{K}_1 = K \hat{i}$ and $\vec{K}_2 = -K \hat{i}$ in the opposite directions, as shown in the figure below (The extent of the sheets in the y direction is infinite.) Note that K is the current per unit width perpendicular to the flow.



- Find the magnetic field everywhere due to \vec{K}_1 .
- Find the magnetic field everywhere due to \vec{K}_2 .
- Find the magnetic field everywhere due to both current sheets.
- How would your answer in (c) change if both currents were running in the same direction, with $\vec{K}_1 = \vec{K}_2 = K \hat{i}$?

Solution:

- Find the magnetic field everywhere due to \vec{K}_1 .



Consider the Ampere's loop shown above. The enclosed current is given by

$$I_{\text{enc}} = \int \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}} = Kl$$

Applying Ampere's law, the magnetic field is given by

$$B(2l) = \mu_0 Kl \quad \text{or} \quad B = \frac{\mu_0 K}{2}$$

Therefore,

$$\vec{\mathbf{B}}_1 = \begin{cases} -\frac{\mu_0 K}{2} \hat{\mathbf{j}}, & z > \frac{d}{2} \\ \frac{\mu_0 K}{2} \hat{\mathbf{j}}, & z < \frac{d}{2} \end{cases}$$

(b) Find the magnetic field everywhere due to $\vec{\mathbf{K}}_2$.

The result is the same as part (a) except for the direction of the current:

$$\vec{\mathbf{B}}_2 = \begin{cases} \frac{\mu_0 K}{2} \hat{\mathbf{j}}, & z > -\frac{d}{2} \\ -\frac{\mu_0 K}{2} \hat{\mathbf{j}}, & z < -\frac{d}{2} \end{cases}$$

(c) Find the magnetic field everywhere due to both current sheets.

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_2 = \begin{cases} \mu_0 K \hat{\mathbf{j}}, & -\frac{d}{2} < z < \frac{d}{2} \\ 0, & |z| > \frac{d}{2} \end{cases}$$

(d) How would your answer in (c) change if both currents were running in the same direction, with $\vec{\mathbf{K}}_1 = \vec{\mathbf{K}}_2 = K \hat{\mathbf{i}}$?

In this case, $\vec{\mathbf{B}}_1$ remains the same but

$$\vec{\mathbf{B}}_2 = \begin{cases} -\frac{\mu_0 K}{2} \hat{\mathbf{j}}, & z > -\frac{d}{2} \\ \frac{\mu_0 K}{2} \hat{\mathbf{j}}, & z < -\frac{d}{2} \end{cases}$$

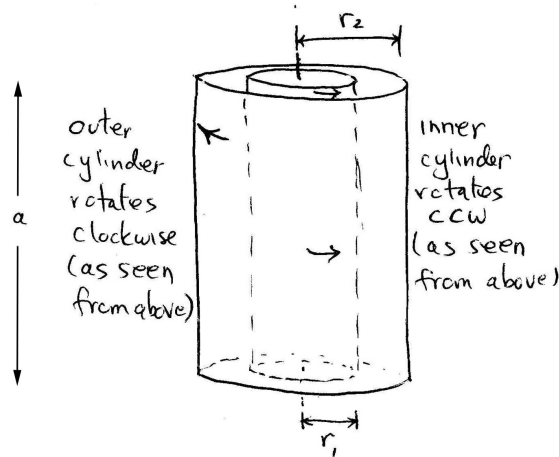
Therefore,

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_2 = \begin{cases} -\mu_0 K \hat{\mathbf{j}}, & z > \frac{d}{2} \\ 0, & -\frac{d}{2} < z < \frac{d}{2} \\ \mu_0 K \hat{\mathbf{j}}, & z < -\frac{d}{2} \end{cases}$$

Finally, to make it twice as hard to turn that means twice as much work, which means that the resistance must be half as much. This is called “loading” the generator – where an increase in load is actually a *decrease* in the resistance.

Problem 9: Rotating Charged Cylinders Ampere's Law

Two very long cylindrical conductors of length a each open at the ends are coaxial and rotating in opposite directions along the coaxial axis with both with angular speed ω . The inner cylinder is rotating in the counterclockwise direction when seen from above. The inner cylinder has radius r_1 and a charge Q_1 distributed uniformly over the surface. The outer cylinder has radius r_2 and a charge Q_2 distributed uniformly over the surface. You may ignore edge effects.



- What are the surface current densities \vec{K}_1 and \vec{K}_2 on the two cylinders?
- Find the magnitude and direction of the magnetic field everywhere, (i) $r < r_1$, (ii) $r_1 < r < r_2$, and (iii) $r > r_2$.

Solution: a) Recall that the surface current density is defined to be $\vec{K} = \sigma \vec{v}$ where σ is the charge per unit area and $\vec{v} = r\omega \hat{\theta}$ is the velocity of the charges on the surface of the cylinder, where $\hat{\theta}$ is a unit vector pointing counterclockwise as seen from above. Therefore for inner cylinder

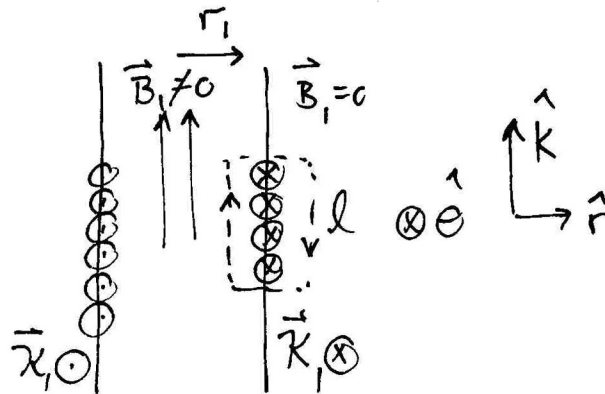
$$\vec{K}_1 = \sigma_1 \vec{v}_1 = \frac{Q_1}{2\pi r_1 a} r_1 \omega \hat{\theta} = \frac{Q_1}{2\pi a} \omega \hat{\theta}$$

and for the outer cylinder

$$\vec{K}_2 = \sigma_2 \vec{v}_2 = -\frac{Q_2}{2\pi r_2 a} r_2 \omega \hat{\theta} = -\frac{Q_2}{2\pi a} \omega \hat{\theta}$$

- We shall now apply Ampere's Law for the each cylinder separately and use the superposition principle to find the magnetic field in the three regions.

First we consider the inner cylinder. Choose coordinates and an Amperian loop shown in the figure below.



The current enclosed is into the plane of the figure hence positive by our choice of circulation direction and is given by

$$I_{\text{enc}} = K_1 l = \frac{Q_1}{2\pi a} \omega l$$

We assume the cylinder is very long so the magnetic field is uniform inside and zero outside like the magnetic field of an infinite solenoid. The line integral of the magnetic field around the closed square loop is therefore

$$\oint \vec{\mathbf{B}}_1 \cdot d\vec{\mathbf{s}} = B_1 l.$$

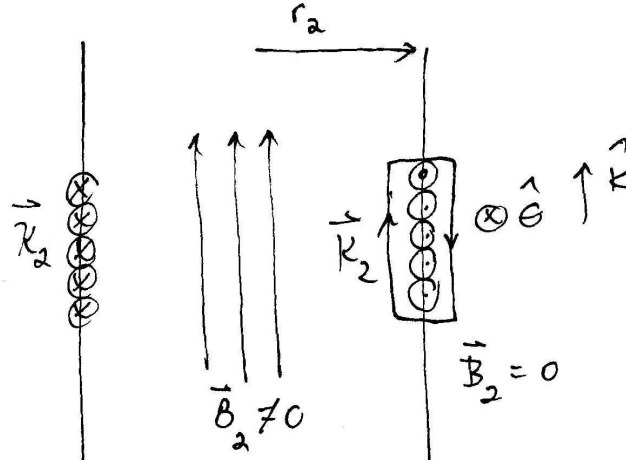
Therefore Ampere's Law $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{\text{enc}}$ becomes

$$B_1 l = \mu_0 \frac{Q_1}{2\pi a} \omega l$$

and so the magnetic field of the inner cylinder points upwards in the positive $\hat{\mathbf{k}}$ -direction and is given by

$$\vec{\mathbf{B}}_1 = \begin{cases} \frac{\mu_0 Q_1 \omega}{2\pi a} \hat{\mathbf{k}}; & r < r_1 \\ \vec{\mathbf{0}}; & r > r_1 \end{cases}$$

We shall now do a similar calculation for the magnetic field due to the outer cylinder with Amperian loop and circulation direction shown in the figure below.



The current enclosed is out the plane of the figure hence negative by our choice of circulation direction and is given by

$$I_{\text{enc}} = -K_2 l = -\frac{Q_2}{2\pi a} \omega l.$$

Therefore the magnetic field due to the outer cylinder points in the opposite direction as the magnetic field from the inner cylinder and using Ampere's law as we just did yields

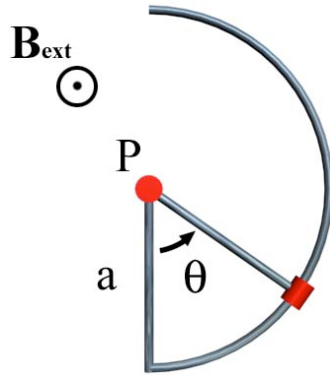
$$\vec{B}_2 = \begin{cases} -\frac{\mu_0 Q_2 \omega}{2\pi a} \hat{k}; & r < r_2 \\ \vec{0}; & r > r_2 \end{cases}.$$

By the superposition principle the magnetic field due to both cylinders is

$$\vec{B}_1 + \vec{B}_2 = \begin{cases} \frac{\mu_0 Q_1 \omega}{2\pi a} (Q_1 - Q_2) \hat{k}; & r < r_1 \\ -\frac{\mu_0 Q_2 \omega}{2\pi a} \hat{k}; & r_1 < r < r_2 \\ \vec{0}; & r > r_2 \end{cases}$$

Problem 10: Generator

A “pie-shaped” circuit is made from a straight vertical conducting rod of length a welded to a conducting rod bent into the shape of a semi-circle with radius a (see sketch). The circuit is completed by a conducting rod of length a pivoted at the center of the semi-circle, (at point P), and free to rotate about that point. This moving rod makes electrical contact with the vertical rod at one end and the semi-circular rod at the other end. The angle θ is the angle between the vertical rod and the moving rod, as shown. The circuit sits in a constant magnetic field \vec{B}_{ext} pointing out of the page.



(a) If the angle θ is increasing with time, what is the direction of the resultant current flow around the “pie-shaped” circuit? What is the direction of the current in the circuit?

Solution: The flux out of the page is increasing, so we want to generate a field into the page (Lenz’ Law). This requires a clockwise current (see arrows beside pie shaped wedge).

For the next two parts, assume that the angle θ is increasing at a constant positive rate, $\omega = d\theta(t) / dt$.

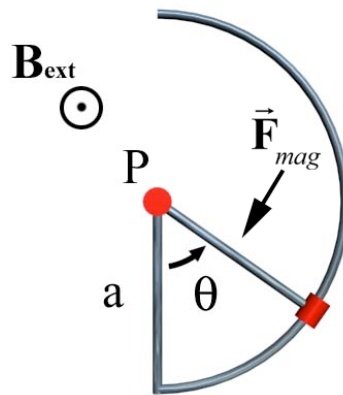
(b) What is the magnitude of the rate of change of the magnetic flux through the “pie-shaped” circuit due to \vec{B}_{ext} only (you may ignore the magnetic field associated with any induced current in the circuit)?

Solution:

$$\frac{d\Phi_B}{dt} = \frac{d}{dt}(B_{\text{ext}}A) = B_{\text{ext}} \frac{d}{dt}\left(\pi a^2 \cdot \frac{\theta}{2\pi}\right) = \frac{B_{\text{ext}} a^2}{2} \frac{d\theta}{dt} = \frac{B_{\text{ext}} a^2}{2} \omega$$

(c) If the “pie-shaped” circuit has a constant resistance R , what is the magnitude and direction of the magnetic force due to the external field on the moving rod in terms of the quantities given.

Solution:



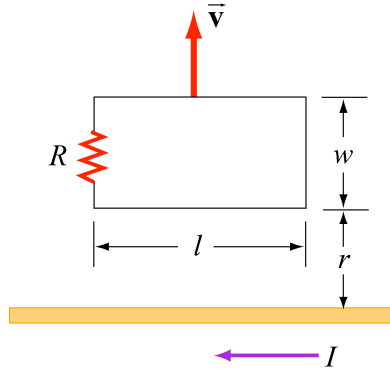
The magnetic force is determined by the current, which is determined by the EMF, which is determined by Faraday's Law:

$$\begin{aligned}\epsilon &= \frac{d}{dt} \Phi_B = \frac{B_{\text{ext}} a^2}{2} \omega \Rightarrow I = \frac{\epsilon}{R} = \frac{B_{\text{ext}} a^2 \omega}{2R} \\ \Rightarrow |\vec{\mathbf{F}}_{\text{mag}}| &= IaB_{\text{ext}} = \frac{B_{\text{ext}}^2 a^3 \omega}{2R}\end{aligned}$$

The force opposes the motion, which means it is currently down and to the left (the cross product of a radially outward current with a magnetic field out of the page).

Problem 11: Moving Loop

A rectangular loop of dimensions l and w moves with a constant velocity \vec{v} away from an infinitely long straight wire carrying a current I in the plane of the loop, as shown in the figure. The total resistance of the loop is R .



(a) Using Ampere's law, find the magnetic field at a distance s away from the straight current-carrying wire.

Consider a circle of radius s centered on the current-carrying wire. Then around this Amperian loop,

$$\oint \vec{B} \cdot d\vec{s} = B(2\pi s) = \mu_0 I$$

which gives

$$B = \frac{\mu_0 I}{2\pi s} \text{ (into the page)}$$

(b) What is the magnetic flux through the rectangular loop at the instant when the lower side with length l is at a distance r away from the straight current-carrying wire, as shown in the figure?

$$\Phi_B = \iint_S \vec{B} \cdot d\vec{A} = \int_r^{r+w} \left(\frac{\mu_0 I}{2\pi s} \right) l ds = \frac{\mu_0 I l}{2\pi} \ln \left(\frac{r+w}{r} \right) \text{ (into the page)}$$

(c) At the instant the lower side is a distance r from the wire, find the induced emf and the corresponding induced current in the rectangular loop. Which direction does the induced current flow?

The induced emf is

$$\mathcal{E} = -\frac{d}{dt} \Phi_B = -\frac{\mu_0 I l}{2\pi} \frac{r}{(r+w)} \left(\frac{-w}{r^2} \right) \frac{dr}{dt} = \frac{\mu_0 I l}{2\pi} \frac{vw}{r(r+w)}$$

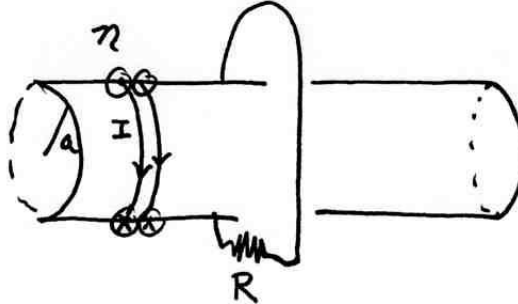
The induced current is

$$I = \frac{|\mathcal{E}|}{R} = \frac{\mu_0 I l}{2\pi R} \frac{vw}{r(r+w)}$$

The flux into the page is decreasing as the loop moves away because the field is growing weaker. By Lenz's law, the induced current produces magnetic fields which tend to oppose the change in magnetic flux. Therefore, the current flows clockwise, which produces a self-flux that is positive into the page.

Problem 12: Solenoid

A long solenoid of radius a carrying n turns per unit length, is looped by a wire of resistance R as shown in the figure below.



If the current $I(t)$ in the solenoid is increasing at a constant rate

$$dI / dt = h = \text{constant}$$

- What is the magnitude of the current that flows through the loop? Indicate the direction of the current in the figure.
- If the current I in the solenoid is constant but the solenoid is pulled out of the loop and reinserted in the opposite direction, what total charge passes through the resistor?
- While the current is changing in the wire that is wrapped around the solenoid, what is the direction and magnitude of the electric field inside the solenoid?

Solutions:

- What is the magnitude of the current that flows through the loop? Indicate the direction of the current in the figure.

Answer: We use the Ampere's Law to calculate the magnetic field

$$\oint_{loop} \vec{B} \cdot d\vec{r} = \mu_0 I_{enc}$$

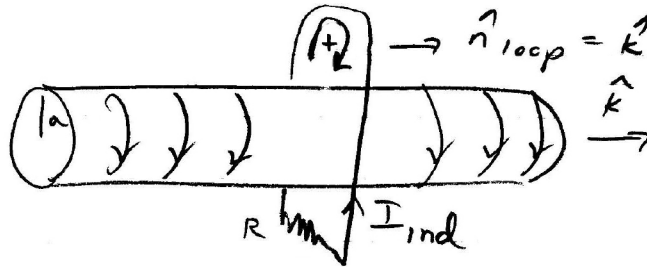
If we choose a rectangular loop of length l , then Ampere's Law becomes

$$Bl = \mu_0 nIl$$

So the magnetic field inside the solenoid is given by

$$\vec{B} = \mu_0 nI \hat{k}$$

where $\hat{\mathbf{k}}$ is a unit vector pointing to the right on the figure below.



The magnetic flux through the solenoid is

$$\Phi_{mag}^{loop} = \iint_{turn} \vec{\mathbf{B}} \cdot d\vec{\mathbf{a}} = B\pi a^2 = \mu_0 \pi a^2 nI.$$

The electromotive force in the loop is

$$\varepsilon = -\frac{d\Phi_{mag}^{loop}}{dt} = -\mu_0 \pi a^2 n \frac{dI}{dt} = -\mu_0 \pi a^2 nh.$$

The induced current has magnitude

$$I_{ind} = \frac{|\varepsilon|}{R} = \frac{\mu_0 \pi a^2 nh}{R}.$$

The current flows opposite the orientation as shown in the figure.

- b) If the current I in the solenoid is constant but the solenoid is pulled out of the loop and reinserted in the opposite direction, what total charge passes through the resistor?

Answer: Suppose the solenoid is pulled out of the loop and reinserted in the opposite direction in a time interval $\Delta t = t_f - t_0$. The change in magnetic flux through the loop is then

$$\Delta\Phi_{mag}^{loop} = \Phi_{mag}^{loop}(t_f) - \Phi_{mag}^{loop}(t_0) = -B\pi a^2 - B\pi a^2 = -\mu_0 2\pi a^2 nI.$$

The induced current is then

$$|I_{ind}| = \frac{\Delta Q}{\Delta t} = \frac{1}{R} \frac{\Delta\Phi_{mag}^{loop}}{\Delta t} = \frac{\mu_0 2\pi a^2 nI}{\Delta t R}.$$

So the total charge passes through the resistor is given by

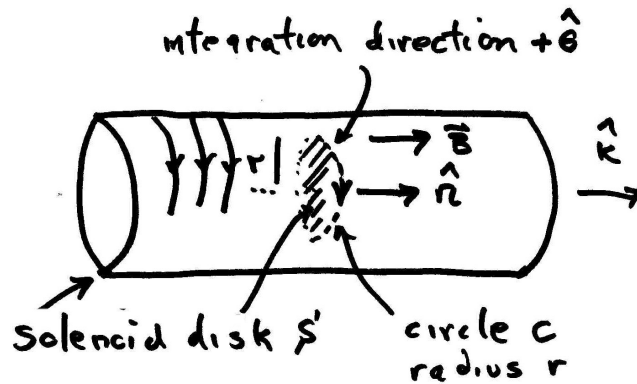
$$\Delta Q = \frac{\mu_0 2\pi a^2 n I}{R}.$$

- c) While the current is changing in the wire that is wrapped around the solenoid, what is the direction and magnitude of the electric field inside the solenoid?

Answer: We can use Faraday's Law inside the solenoid in the form

$$\oint_C \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \iint_S \vec{B} \cdot \hat{n} da.$$

Choose a circle of radius r for the closed path C for the left hand side of Faraday's Law with integration direction shown in the figure below. With these choices $\hat{n} = \hat{k}$.



Then Faraday's Law becomes

$$E_\theta 2\pi r = -\frac{dB_z}{dt} \pi r^2$$

We can solve for the tangential component of the electric field

$$E_\theta = -\frac{1}{2} \frac{dB_z}{dt} r$$

Using our result for the magnetic field of the solenoid

$$E_\theta = -\frac{dB_z}{dt} \frac{r}{2} = -\mu_0 n \frac{r}{2} \frac{dI}{dt}$$

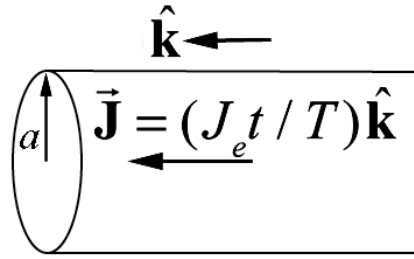
Because $dI / dt = h$, the tangential component of the electric field is given by

$$E_{\theta} = -\frac{dB}{dt} \frac{r}{2} = -\mu_0 n \frac{r}{2} h.$$

Because $E_{\theta} < 0$, it is pointing in the negative- $\hat{\boldsymbol{\theta}}$ direction and thus

$$\vec{\mathbf{E}} = -\mu_0 n \frac{r}{2} h \hat{\boldsymbol{\theta}}.$$

Problem 13 Ampere's Law and Faraday's Law



Consider an infinite cylindrical solid wire that has radius a . The wire has a time varying current with the current density as a function of time given by the following expression:

$$\vec{J} = \begin{cases} \vec{0}; & t \leq 0 \\ (J_e t / T) \hat{k}; & 0 \leq t \leq T \\ J_e \hat{k}; & T \leq t \end{cases}$$

where J_e is positive constant with units of amps per square meter.

- a) Find the direction and magnitude of the magnetic field for the interval $0 \leq t \leq T$ in the regions: (i) $0 \leq r \leq a$; (ii) $r \geq a$, where r is the distance from the symmetry axis of the wire.

Clearly show all your work. Answers without justification will receive no credit.

$0 \leq r \leq a$: We take an Amperian loop which is a circle of radius $r < a$. We have

$$\oint_{\text{closed path}} \vec{B} \cdot d\vec{s} = 2\pi r B_\theta = \mu_0 \iint_S \vec{J} \cdot \hat{n} da = \mu_0 \int_0^r 2\pi r' dr' \left(\frac{J_e t}{T} \right) = \mu_0 \pi r^2 \left(\frac{J_e t}{T} \right) \Rightarrow \vec{B} = \frac{r}{2} \mu_0 (J_e t / T) \hat{\theta}$$

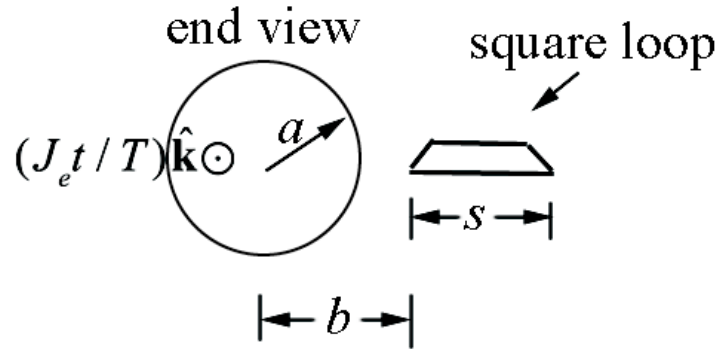
$\hat{\theta}$ is counterclockwise looking from left

$r > a$: We take an Amperian loop which is a circle of radius $r > a$. We have

$$\oint_{\text{closed path}} \vec{B} \cdot d\vec{s} = 2\pi r B_\theta = \mu_0 \iint_S \vec{J} \cdot \hat{n} da = \mu_0 \int_0^a 2\pi r' dr' \left(\frac{J_e t}{T} \right) = \mu_0 \pi a^2 \left(\frac{J_e t}{T} \right) \Rightarrow \vec{B} = \frac{a^2}{2r} \mu_0 (J_e t / T) \hat{\theta}$$

$\hat{\theta}$ is counterclockwise looking from left

- b) Suppose a square conducting loop with resistance R , and side s is placed in the region $r > a$, such that the nearest side of the loop to the wire is a distance b from the axis as shown in the end view figure below. What is the induced current in the square loop for the time interval $0 \leq t \leq T$? Draw the direction of the induced current on the figure.



The direction of the current is clockwise as viewed from above. In the time interval $0 \leq t \leq T$, we have

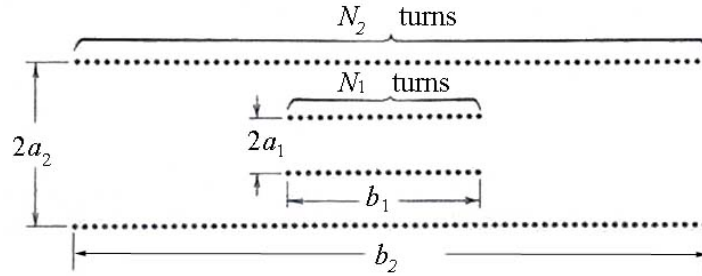
$$\begin{aligned} \left| \frac{d\Phi}{dt} \right| &= \left| \frac{d}{dt} \iint_{\text{open surface } S} \vec{B} \cdot \hat{n} \, da \right| = \left| \frac{d}{dt} \int_b^{b+s} dz \int_r^r \frac{a^2}{2r} \mu_0 (J_e t / T) \right| \\ &= \frac{sa^2}{2} \mu_0 (J_e t / T) \int_b^{b+s} \frac{dr}{r} = \frac{sa^2}{2} \mu_0 (J_e t / T) \ln \left(\frac{b+s}{b} \right) \\ |I| &= \frac{1}{R} \left| \frac{d\Phi}{dt} \right| = \mu_0 \frac{sa^2 J_e}{2RT} \ln \left(\frac{b+s}{b} \right) \end{aligned}$$

- c) What is the direction and magnitude of the force due to the induced current on the square loop during the time interval $0 \leq t \leq T$? What is the direction and magnitude of the torque due to the induced current on the square loop during the time interval $0 \leq t \leq T$?

The force on the far side and the near side of the loop are equal and opposite, and cancel. The force on the left side of the loop is to the right and the force on the right side is to the left, giving a net force pushing the loop away from the wire. If \hat{x} is to the right, then

$$\vec{F} = \hat{x} s I \left[B \Big|_{r=b+s} - B \Big|_{r=b} \right] = \hat{x} s I \left(\frac{a^2}{2} \mu_0 (J_e t / T) \right) \left(\frac{1}{b} - \frac{1}{b+s} \right) \text{ where } |I| = \mu_0 \frac{sa^2 J_e}{2RT} \ln \left(\frac{b+s}{b} \right)$$

Problem 14: Mutual Inductance. The figure shows a solenoid of radius a_1 and length b_1 located inside a longer solenoid of radius a_2 and length b_2 . The total number of turns is N_1 on the inner coil, N_2 on the outer.



- What is the mutual inductance M .
- If a current I_1 flows through the little solenoid, what is the total flux through the big one?

Solution: Let solenoid 2 carry a current I . The the magnetic field B_2 inside solenoid 2 is given by

$$B_2 = \mu_0 n_2 I$$

where n_2 is the number of turns per unit length, i.e. $n_2 = N_2 / b_2$. The flux of B_2 through solenoid 1 is

$$\Phi_{12} = N_1 B_2 \pi a_1^2 = N_1 n_2 \mu_0 I \pi a_1^2 = N_1 N_2 \mu_0 I \pi a_1^2 / b_2.$$

The emf in solenoid 1 is then

$$\epsilon_1 = -\frac{d\Phi_{12}}{dt} = -\frac{N_1 N_2 \mu_0 \pi a_1^2}{b_2} \frac{dI}{dt} = -M \frac{dI}{dt}$$

Therefore the coefficient of mutual inductance is

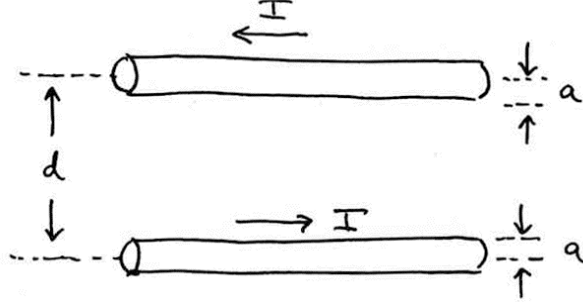
$$M = \frac{N_1 N_2 \mu_0 \pi a_1^2}{b_2}$$

- If a current I_1 flows through the little solenoid, what is the total flux through the big one?

Solution:

$$\Phi_{21} = M I_1 = \frac{N_1 N_2 \mu_0 \pi a_1^2}{b_2} I_1.$$

Problem 15 Self-Inductance of Two Wires Two long parallel wires, each of radius a , whose centers are a distance d apart carry equal currents in opposite directions. What is the self-inductance of a length l of such a pair of wires? You may neglect the flux within the wires themselves.



Solution: The magnetic field of both wires points out of the plane of the figure above for a point P located between the wires a distance y from the center of the lower wire. The magnetic field at P is the superposition of the two contributions. We can use Ampere's Law to find the magnetic field at the point P due to both wires,

$$\vec{\mathbf{B}}_{\text{lower}}(y) = \frac{\mu_0 I}{2\pi y} \hat{\mathbf{k}}$$

$$\vec{\mathbf{B}}_{\text{upper}}(y) = \frac{\mu_0 I}{2\pi(d-y)} \hat{\mathbf{k}}.$$

Hence

$$\vec{\mathbf{B}}(y) = \vec{\mathbf{B}}_{\text{upper}}(y) + \vec{\mathbf{B}}_{\text{lower}}(y) = \frac{\mu_0 I}{2\pi(d-y)} \hat{\mathbf{k}} + \frac{\mu_0 I}{2\pi y} \hat{\mathbf{k}}$$

We now want to calculate the magnetic flux through a small rectangle of length l and width $d - 2a$. We choose for the area element $d\vec{\mathbf{a}} = l dy \hat{\mathbf{k}}$. Then the magnetic flux is

$$\begin{aligned} \Phi &= \int_{\text{loop}} \vec{\mathbf{B}}(y) \cdot d\vec{\mathbf{a}} = \int_{y=a}^{y=d-a} \left(\frac{\mu_0 I}{2\pi(d-y)} \hat{\mathbf{k}} + \frac{\mu_0 I}{2\pi y} \hat{\mathbf{k}} \right) \cdot l dy \hat{\mathbf{k}} \\ &= \frac{\mu_0 I l}{2\pi} \int_{y=a}^{y=d-a} \left(\frac{1}{d-y} + \frac{1}{y} \right) dy = \frac{\mu_0 I l}{2\pi} \left(-\ln(d-y) \Big|_{y=a}^{y=d-a} + \ln(y) \Big|_{y=a}^{y=d-a} \right) \\ &= \frac{\mu_0 I l}{2\pi} \left((-\ln(a) + \ln(d-a)) + (\ln(d-a) - \ln(a)) \right) = \frac{\mu_0 I l}{\pi} \ln \left(\frac{d-a}{a} \right) \end{aligned}$$

The self-induction of a length l is then

$$L = \frac{\Phi}{I} = \frac{\mu_0 l}{\pi} \ln\left(\frac{d-a}{a}\right).$$

Problem 16: Stored Magnetic Energy

- a) A magnetic field exists in most of interstellar space in our galaxy. There is evidence that its strength in most regions is between 10^{-11} T and 10^{-10} T. Adopting 3×10^{-11} T as a typical value, find, in order of magnitude, the total energy stored in the magnetic field of the galaxy. For this purpose you may assume that the galaxy is a disk roughly 10^{21} m in diameter and 10^{19} m thick.
- b) What is the magnetic energy density of the earth's magnetic field of 5.0×10^{-5} T?
- c) Assuming the magnetic field is relatively constant over distances small compared with the earth's radius and neglecting the variations near the earth's magnetic poles, how much energy would be stored in a shell between the earth's surface and 15 km above the surface? (Look up any quantities that you may need.)

Solution:

- a) Because we are approximating that the magnetic field is uniform, and modeling the galaxy as a flat disk, the energy stored in the magnetic field is

$$U = \frac{1}{2\mu_0} \int_{\text{disk}} B^2 dV \approx \frac{B^2}{2\mu_0} \pi R^2 s$$

where $B \approx 3 \times 10^{-11}$ T, $R \approx 0.5 \times 10^{21}$ m and $s \approx 10^{19}$ m. Then

$$U \approx \frac{(3 \times 10^{-11} \text{ T})^2}{2(4\pi \times 10^{-7} \text{ T}^2 \cdot \text{m}^3 \cdot \text{J}^{-1})} \pi (0.5 \times 10^{21} \text{ m})^2 (10^{19} \text{ m}) = 0.3 \times 10^{48} \text{ J} \approx 10^{47} \text{ J}$$

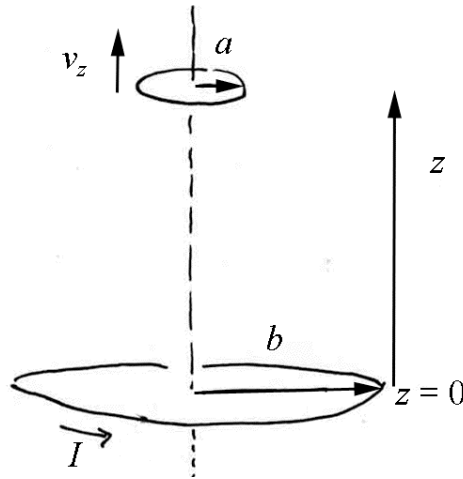
- b) The magnetic energy density of the earth is

$$U = \frac{B^2}{2\mu_0} = \frac{(5.0 \times 10^{-5} \text{ T})^2}{(4\pi \times 10^{-7} \text{ T}^2 \cdot \text{m}^3 \cdot \text{J}^{-1})} = 2.0 \times 10^{-3} \text{ J} \cdot \text{m}^3$$

- c) The mean radius of the earth is $R = 6.4 \times 10^6$ m and so the magnetic energy stored in a shell between the earth's surface and $s = 15 \times 10^3$ m above the surface is approximately

$$U \approx \frac{B^2}{2\mu_0} 4\pi R^2 s = (2.0 \times 10^{-3} \text{ J} \cdot \text{m}^3) (4\pi) (6.4 \times 10^6 \text{ m})^2 (15 \times 10^3 \text{ m}) = 1.5 \times 10^{16} \text{ J}$$

Problem 17 Challenge (harder than exam 3 level): The figure below shows two loops of wire having the same axis. The smaller loop has radius a and resistance R and the larger loop has radius b . The smaller loop is above the larger one, by a distance z , which is large compared to the radius b of the larger loop, ($z \gg b$). Hence with current I through the larger loop as indicated, the consequent magnetic field is nearly constant through the plane area bounded by the smaller loop. Suppose now that z is not constant but is changing at the positive constant rate $v_z = dz/dt > 0$ (z increasing).



- Determine the magnetic flux across the area bounded by the smaller loop as a function of z .
- Compute the emf generated in the smaller loop and the induced current at the instant the loop is located at z and moving with z -component of velocity $v_z = dz/dt > 0$. Determine the direction of the induced current in the smaller loop.
- What is the induced force on the small ring at the instant the loop is located at z and moving with z -component of velocity $v_z = dz/dt > 0$?

Solution:

- We begin by calculating the magnetic field at the center of the smaller upper loop due to the lower loop using the Biot Savart Law

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{Id\vec{s} \times \hat{r}}{r^2}.$$

Choose cylindrical coordinates with unit vectors $(\hat{\rho}, \hat{\theta}, \hat{k})$. Note $\hat{\rho}$ is the unit vector that points radially outward in the plane. We have for the current element $Id\vec{s} = Ib d\theta \hat{\theta}$, the distance from the current element to the center of the upper ring is $r = (z^2 + b^2)^{1/2}$, and the

unit vector from the current element to the field point at the center of the rings is $\hat{\mathbf{r}} = (z\hat{\mathbf{k}} - b\hat{\boldsymbol{\rho}}) / (z^2 + b^2)^{1/2}$. Therefore the Biot-Savart Law becomes

$$d\bar{\mathbf{B}}_1 = \frac{\mu_0}{4\pi} \frac{Id\bar{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi} \frac{(IR_2 d\theta \hat{\boldsymbol{\theta}} \times (z\hat{\mathbf{k}} - b\hat{\boldsymbol{\rho}}))}{(z^2 + b^2)^{3/2}} = \frac{\mu_0}{4\pi} \frac{IR_2 d\theta (x\hat{\boldsymbol{\rho}} + b\hat{\mathbf{k}})}{(z^2 + b^2)^{3/2}}$$

Integrating around the larger ring the non-zero contribution is

$$\bar{\mathbf{B}}_1 = \frac{\mu_0 Ib^2}{2(z^2 + b^2)^{3/2}} \hat{\mathbf{k}}.$$

Because the small loop is very far away from the large loop we can approximate the magnetic flux through the small loop by

$$\Phi \simeq \frac{\mu_0 Ib^2}{2(z^2 + b^2)^{3/2}} \pi a^2.$$

b) We can use the chain rule to calculate the change in magnetic flux

$$\frac{d\Phi}{dt} \simeq \frac{d\Phi}{dz} \frac{dz}{dt} = -\frac{3\mu_0 Ib^2 \pi a^2 z}{2(z^2 + b^2)^{5/2}} v_z.$$

Now we use Faraday's Law to calculate the emf at $z = \alpha b$

$$\mathcal{E} = \frac{3\mu_0 Ib^2 \pi a^2 z}{2(z^2 + b^2)^{5/2}} v_z.$$

The induced current is therefore

$$I_{ind} = \frac{\mathcal{E}}{R} = \frac{3\mu_0 Ib^2 \pi a^2 z}{2(z^2 + b^2)^{5/2} R} v_z.$$

We chose a unit normal for the upper loop in the positive $\hat{\mathbf{k}}$ -direction corresponding to a counterclockwise circulation direction (as seen from above). Because the emf is positive this means that the induced current is in the counterclockwise direction as seen from above. We also note that the magnetic field is up, the flux is up and decreasing as the small loop moves upward, therefore we need an induced magnetic field pointing upwards which corresponds to an induced current counterclockwise as seen from above.

c) The small ring now acts like a magnetic dipole with magnetic moment

$$\vec{\mu} = \frac{3\mu_0 I b^2 \pi a^2 z}{2(z^2 + b^2)^{5/2} R} v_z \pi a^2 (-\hat{\mathbf{k}}),$$

where the z -component of the magnetic dipole moment is given by

$$\mu_z = -\frac{3\mu_0 I b^2 \pi^2 a^4 z v_z}{2(z^2 + b^2)^{5/2} R}$$

The z -component of the force on the upper ring is then

$$F_z = \mu_z \frac{\partial B_z}{\partial z}.$$

From our calculation in part a), the partial derivative of the magnetic field at

$$\frac{\partial B_z}{\partial z} = -\frac{3\mu_0 I b^2 z}{2(z^2 + b^2)^{5/2}}$$

Therefore the z -component of the force on the upper ring is then

$$F_z = \mu_z \frac{\partial B_z}{\partial z} = \left(-\frac{3\mu_0 I b^2 \pi^2 a^4 z v_z}{2(z^2 + b^2)^{5/2} R} \right) \left(-\frac{3\mu_0 I b^2 z}{2(z^2 + b^2)^{5/2}} \right) = \left(\frac{9\mu_0^2 I^2 b^4 \pi^2 a^4 z^2 v_z}{4(z^2 + b^2)^5 R} \right) > 0.$$