Cycle structure and pattern avoidance of *abc*-permutations

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July 30, 2010

abc-permutations

Definition

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Examples

[67851234]
[67812345]

Work of Pak and Redlich

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Theorem (Pak-Redlich, 2008)

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Let $\mathbf{p}(n)$ be the probability that an abc-permutation of length n is a long cycle. Then

$$\lim_{n\to\infty} \mathbf{p}(n) = \frac{6}{\pi^2}.$$

Cycle structure of *abc*-permutations

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Theorem

Let $k = \gcd(a+b, b+c)$*. Then* σ_{abc} *is composed of* k *cycles of lengths* $\lfloor \frac{n}{k} \rfloor$ *and* $\lceil \frac{n}{k} \rceil$*. These cycles correspond to the residue classes of* n *modulo* k*.*

Cycle structure statistics of *abc*-permutations

We investigate the probability $\mathbf{p}_k(n)$ that a random *abc*-permutation of length n has exactly k cycles.

Theorem

We have

$$\lim_{n\to\infty} \mathbf{p}_k(n) = \frac{1}{k^2}\frac{6}{\pi^2}.$$

Ideas from the classification of cycle structure

First, note that if $x \leq c$, we have

$$\sigma_{abc}(x) = x + a + b.$$

For ease of notation, let $d = c - a$. If $c < x \leq b + c$, we have

$$\sigma_{abc}(x) = x - d,$$

and if $x > b + c$, we have

$$\sigma_{abc}(x) = x - a - b.$$

Ideas from the classification of cycle structure, cont

Therefore, for all i and x we have

$$\sigma_{abc}^{i}(x) = x + m(a+b) - ld.$$

for some integers m, l satisfying $|m| + |l| \leq i$.

Ideas from the classification of cycle structure, cont

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for some integers m, l satisfying $|m| + |l| \leq i$.

Let $k = \gcd(a+b, d) = \gcd(a+b, b+c)$. Then the orbit of x under σ_{abc} will stay within one residue class modulo k.

Ideas from the classification of cycle structure, cont

To show that each residue class contains exactly one residue class, we note that if x is in a cycle of length j, then

$$\sigma_{abc}^{j}(x) = x + m(a+b) - ld = x.$$

and $m + l \leq j$.

Ideas from the classification of cycle structure, cont

To show that each residue class contains exactly one residue class, we note that if x is in a cycle of length j, then

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and $m + l \leq j$.
We show that $\lceil \frac{n}{k} \rceil / 2 < j$.

Idea of the construction of $\mathbf{p}_k(n)$

Let A_k be the event that $k|(a+b)$ and $(b+c)$, and let B_k be the event that $(a+b)/k$ and $(b+c)/k$ are relatively prime integers. Then

$$\mathbf{p}_k(n) = Pr(A_k) \cdot Pr(B_k \text{ given } A_k)$$

Idea of the construction of $\mathbf{p}_k(n)$, cont

Let $f(n,k)$ be the probability that $k \mid (a+b)$ and $(b+c)$ for a random *abc*-permutation of length n.

Lemma (Pak-Redlich, 2008)

$$f(n,k) = \begin{cases} \frac{\left(\frac{n}{k}+1\right)\left(\frac{n}{k}+2\right)}{(n+1)(n+2)} & \text{if } k \mid n \\ \frac{\lfloor\frac{n}{k}\rfloor\left(\lfloor\frac{n}{k}\rfloor+1\right)}{n(n+1)} & \text{if } k \nmid n. \end{cases}$$

Idea of the construction of $\mathbf{p}_k(n)$, cont

Then

$$\mathbf{p}_k(n) = f(n,k) \cdot Pr(B_k \text{ given } A_k).$$

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If $n \mid k$, we have that $k \mid (a+b)$ and $(b+c)$ implies that $k \mid a, b, c$. Then choosing the blocks a, b, c partitioning n is equivalent to choosing blocks $a/k, b/k, c/k$ partitioning n/k.

Idea of the construction of $\mathbf{p}_k(n)$, cont

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Then $Pr(B_k \text{ given } A_k) = \mathbf{p}(n/k)$, so

$$\mathbf{p}_k(n) = f(n,k)\mathbf{p}(n/k).$$

Definition of $\mathbf{p}_k(n)$

Since

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$$\mathbf{p}_k(n) = \begin{cases} f(n,k)(1 - \sum_{s=2}^{n/k} \mu(s) f(n/k, s)) & \text{if } n \mid k \\ f(n,k)(1 - \sum_{s=2}^{n/k} \mu(s) \Gamma(n/k, s)) & \text{if } n \nmid k \end{cases}$$

where $\Gamma(n,k)$ *is a similar function to* $f(n,k)$.

Limit of $\mathbf{p}_k(n)$

Theorem

We have

$$\lim_{n\to\infty} \mathbf{p}_k(n) = \frac{1}{k^2}\frac{6}{\pi^2}.$$

Pattern avoidance

Let $\sigma \in S_n$, and $\phi \in S_k$ where $k \leq n$.

Definition

We say that σ contains ϕ if there exists a subsequence of σ which is order isomorphic to ϕ. Otherwise, we say that σ avoids ϕ.

Pattern avoidance and *abc*-permutations

Theorem

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In order to prove this theorem we make use of the fact that *abc*-permutations are *reverse layered permutations* with three or fewer layers.

Reverse layered permutations

Definition

A permutation ϕ is called reverse layered if it is of the form $q_1 q_2 \ldots q_k$, where the q_i are strings of consecutively increasing numbers and $q_i > q_j$ if $i < j$.

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Lemma (Monsour, 2002)

The set of reverse layered permutations is the set of 132 *and* 213 *avoiding permutations.*

Proof of classification of *abc*-permutations

Assume that σ is reverse layered and avoids 4321. Then it has at most three layers, so it is an *abc*-permutation.

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Assume that σ is reverse layered and avoids 4321. Then it has at most three layers, so it is an *abc*-permutation.

Now assume that σ is an *abc*-permutation. Then it is reverse layered, so it avoids 132, and 213. It also has at most three layers, so it avoids 4321.

Avenues for further research

1. What is the cycle structure of layered permutations?

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1. What is the cycle structure of layered permutations?
2. What is the structure and number of permutations which avoid 132, 213, and another permutation of length at least four?