First-Order Linear Differential Equations

Linear equations are probably the most important class of differential equations. They will be the main focus of this course.

Definition. The general **first-order linear ODE** has the form:

$$A(t)\frac{dx}{dt} + B(t)x(t) = C(t). \tag{1}$$

We'll see that we often need to put it in the form:

$$\frac{dx}{dt} + p(t)x(t) = q(t) \tag{2}$$

We'll call (**??**) **standard form**. We can always convert (**??**) to standard form by dividing by $A(t)$.

1 Terminology and Notation

The functions $A(t)$, $B(t)$ in (**??**) and $p(t)$ in (**??**) are called the **coefficients** of the ODE. If A and B (or p) are constant we say the equation is a **constant coefficient** DE.

We use the familiar notations x' or $\dot{x}$ for the derivative of x. With some exceptions, we'll use $\dot{x}$ to mean the derivative with respect to time and x' for other types of derivatives.

2 Homogeneous/Inhomogeneous

If $C(t) = 0$ in (**??**) the resulting equation:

$$A(t)\dot{x} + B(t)x = 0$$

is called **homogeneous**[1]. Likewise for $\dot{x} + p(t)x = 0$.

[1] Homogeneous is not the same as homogenous (or homogenized). The syllable "ge" has a long e and is stressed in homogeneous, while the syllable "mo" is stressed in homogenous.

Otherwise the equation is **inhomogeneous**.

In the next session we will see a general analytic method for solving first-order linear ODE's. For now, note that if A, B and C are *constant* then the equation is separable:

$$\frac{A\,dx}{C - Bx} = dt.$$

3 Examples

We start with two examples that are modeled by first-order linear ODE's.

Example 1. In session 1 we modeled an oryx population x with natural growth rate k and harvest rate h:

$$\dot{x} = kx - h, \text{ or } \dot{x} - kx = -h.$$

Double check sesssion number. – HB

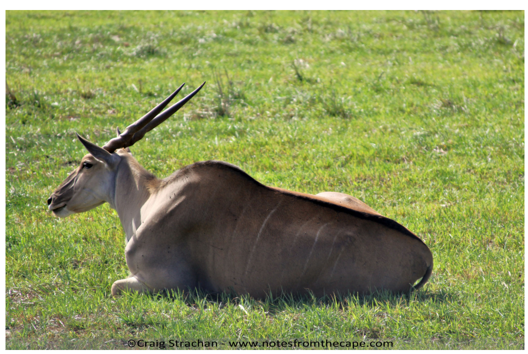

Figure 1: Oryx. Image courtesy of Cape Town Craig on flickr.

These examples doen't have "solutions" as such. – HB We repeat the argument leading to this model. We start with the population $x(t)$ at time t. A natural growth rate k means that after a short time Δt we would expect there to be approximately $kx(t)\Delta t$ more oryx. However, in that same time $h\Delta t$ oryx are harvested. So we have the net change in the oryx population:

$$\Delta x \approx kx(t)\Delta t - h\Delta t \qquad \Longrightarrow \qquad \frac{\Delta x}{\Delta t} \approx kx(t) - h.$$

Now, letting the time interval Δt approach 0 we get the ODE $\frac{dx}{dt} = kx(t) - h$.

Note: if the rates k and h vary with time, the modeling process will lead to the same differential equation:

$$\frac{dx}{dt} = k(t)x(t) - h(t) \text{ or } \frac{dx}{dt} - k(t)x(t) = -h(t).$$

Example 2. Bank account: I have a bank account. It has $x(t)$ dollars in it. x is a function of time. I can deposit money in the account and make withdrawals from it. The bank pays me rent for the money in my account. This is called interest.

In the old days a bank would pay interest at the end of the month on the balance at the beginning of the month. We can model this mathematically.

With $\Delta t = 1/12$, the statement at the end of the month will read:

$$x(t + \Delta t) = x(t) + Ix(t)\Delta t + [\text{deposits} - \text{withdrawals between } t \text{ and } t + \Delta t].$$

I has units $(\text{year})^{-1}$. These days I is typically very small, say $1\% = 0.01$. You don't get 1% each month! you get 1/12 of that.

You can think of a withdrawal as a negative deposit, so I will call everything a deposit.

Nowadays interest is usually computed daily. This is a step on the path to the enlightenment afforded by calculus, in which $\Delta t \to 0$.

In order to reach enlightenment, I want to record deposits minus withdrawals as a *rate*, in dollars per year. Suppose I contribute \$100 sometime every month, and make no withdrawals. My total deposits up to time t – my "cumulative total" deposit $Q(t)$ – has a graph like the one in Figure **??**.

In keeping with letting $\Delta t \to 0$, we should imagine that I am making this contribution continually at the constant rate of \$1200/year. Then the graph of $Q(t)$ is a straight line with slope 1/1200, shown in Figure **??**. The derivative $Q'(t) = q(t)$ is constant.

In general, say I deposit at the rate of $q(t)$ dollars per year. The value of $q(t)$ might vary over time, and might be negative from time to time, because withdrawals are merely negative deposits.

```
   Q
      |             ____|
      |         ___|
      |      __|
      |__|___________________________
          |   |   |   |   |   |   |     t
```

Figure 2: With periodic deposits to a bank account, the graph of $Q(t)$ is a step function.

```
   Q
      |             .
      |         .
      |      .
      |______________________________
          |   |   |   |                  t
```

Figure 3: With continuous deposits to a bank account, the graph of $Q(t)$ is a straight line.

So (assuming $q(t)$ is continuous),

$$x(t + \Delta t) \approx x(t) + Ix(t)\Delta t + q(t)\Delta t.$$

Now subtract $x(t)$ and divide by Δt:

$$\frac{x(t + \Delta t) - x(t)}{\Delta t} \approx Ix + q$$

Next, let the interest period Δt tend to zero:

$$\dot{x} = Ix + q.$$

Note: $q(t)$ can certainly vary in time. The interest rate can too. In fact the interest rate might depend upon x as well: a larger account will probably earn a better interest rate. Neither feature affects the derivation of this equation, but if I does depend upon x as well as t , then the equation we are looking at is no longer linear. So let's say $I = I(t)$ and $q = q(t)$.

We can put the ODE into standard form:

$$\dot{x} - Ix = q.$$

Each symbol represents a function of t.