Superposition Principle

1 Superposition Principle for Inputs

One of the most important properties of first-order linear equations is the **superposition principle**. In fact, we will see that superposition is the defining characteristic of linear equations or any order.

The superposition principle allows us to break up a problem into simpler problems and then at the end assemble the answer from its simpler pieces. For the ODE $y' + p(t)y = q(t)$, let q_1 and q_2 be inputs and c_1 and c_2 be constants. Then:

$$q_1 \to y_1, q_2 \to y_2 \Rightarrow c_1q_1 + c_2q_2 \to c_1y_1 + c_2y_2.$$

This is true because the ODE is linear. The proof takes two lines:

$$\begin{aligned}(c_1y_1 + c_2y_2)' + p(c_1y_1 + c_2y_2) &= (c_1y_1' + pc_1y_1) + (c_2y_2' + pc_2y_2)\\ &= c_1q_1 + c_2q_2.\end{aligned}$$

We present some easy examples below.

2 Examples

We will learn later how to find the following solutions. For this example you can check the following solutions by substitution:

i. $\dot{x} + 2x = 1$ has a solution $x(t) = \frac{1}{2}$

ii. $\dot{x} + 2x = e^{-2t}$ has a solution $x(t) = te^{-2t}$

iii. $\dot{x} + 2x = 0$ has a solution $x(t) = e^{-2t}$.

Using the solutions above as a basis, we can solve more complicated equations.

Example 1. Use superposition to find a solution to $\dot{x} + 2x = 1 + e^{-2t}$

Solution. The input is a superposition of the inputs from (i) and (ii). Therefore a solution is $x(t) = \frac{1}{2} + te^{-2t}$.

Example 2. Find a solution to $\dot{x} + 2x = 2 + 3e^{-2t}$.

Solution. The input is $2 \cdot (1) + 3 \cdot (e^{-2t})$; it is a superposition (with coefficients) of the inputs from (i) and (ii). Therefore, $x(t) = 2 \cdot \frac{1}{2} + 3(te^{-2t}) = 1 + 3te^{-2t}$ is a solution.

Example 3. Find *lots* of solutions to $\dot{x} + 2x = 1$

Solution. We can write the input as:

$$1 = 1 + c \cdot 0.$$

That is, as a superposition of the input from (i) and the homogeneous equation (iii). Therefore $x(t) = \frac{1}{2} + ce^{-2t}$ is a solution for any value of the parameter c.

Example 4. Find *lots* of solutions to $\dot{x} + 2x = 1 + e^{-2t}$.

Solution. Use superposition: $x(t) = \frac{1}{2} + te^{-2t} + ce^{-2t}$.