Linear First Order ODE's

1 First Order Linear Equations

Definition: A **linear ODE** is one that can be put in the form:

$$r(t)\dot{x} + p(t)x = q(t), \qquad x = x(t).$$

Here $r(t)$ and $p(t)$ are the *coefficients* of the ODE. The left hand side represents the *system* and the right hand side arises from an *input signal*. A solution $x(t)$ is a *system response* or *output signal*.

We can always divide through by $r(t)$ to get an equation of the standard form:

$$\dot{x} + p(t)x = q(t), \qquad x = x(t). \tag{1}$$

2 Homogeneous equations

The equation is *homogeneous* if q is the *null signal* $q(t) = 0$. This corresponds to letting the system evolve in isolation:

- In the bank example, no deposits and no withdrawals.
- In the RC example, the power source is not providing any voltage increase.

The homogeneous linear equation:

$$\dot{x} + p(t)x = 0 \tag{2}$$

is separable. We can find the solution as follows:

- Separate: $\dfrac{dx}{x} = -p(t)dt.$

- Integrate: $\ln|x| = -\int p(t)dt + c.$

- Exponentiate: $|x| = e^c e^{-\int p(t)dt}.$

- Eliminate the absolute value and reintroduce the lost solution: $x = Ce^{-\int p(t)dt}.$

Example. $\dot{x} + 2tx = 0$

- Separate: $\frac{dx}{x} = -2tdt.$

- Integrate: $\ln|x| = -\int 2tdt = -t^2 + c.$

- Exponentiate: $|x| = e^c e^{-t^2}.$

- Eliminate the absolute value and reintroduce the lost solution: $x = Ce^{-t^2}.$

In the example, we chose a particular anti-derivative of $-2t$, namely $-t^2$. That is what I have in mind to do in general; the constant of integration is taken care of by the constant C.

The **general solution** to (**??**) has the form Cx_h, where x_h is *any* nonzero solution:

$$x_h = e^{-\int p(t)dt}, \qquad x = Cx_h.$$

Below, we see that the inhomogeneous equation (**??**) can be solved by an algebraic trick that produces a sequence of two integrations.

3 Inhomogeneous DE's via Integrating Factors

This method is based on the product rule for integration:

$$\frac{d}{dt}(ux)' = u\dot{x} + \dot{u}x.$$

Start with equation (**??**):

$$\dot{x} + p(t)x = q(t).$$

In order to apply the product rule, we want the sum on the left hand side of the equation to have the form $u\dot{x} + \dot{u}x$ for some function $u(t)$. At present that's not true unless $p(t) = t$. We adjust the equation by multiplying both sides by some function $u(t)$, whose value we will determine later:

$$u\dot{x} + upx = uq. \tag{3}$$

There may be (and will be) many functions u for which the left hand side of this equation is $\frac{d}{dt}(ux)$; we only need to find one of them.

$$\begin{aligned} \frac{d}{dt}(ux) = u\dot{x} + \dot{u}x &= u\dot{x} + upx \\ \dot{u} &= up. \end{aligned}$$

This is separable:

$$\frac{du}{u} = p(t)\,dt$$

and so:

$$\begin{aligned} \ln|u| &= \int p(t)\,dt \\ u &= e^{\int p\,dt}. \end{aligned}$$

Any choice of antiderivative for $p(t)$ will do - we are just looking for one u that works and don't need the general solution.

Now replace the left-hand side of (**??**) by $\frac{d}{dt}(ux)$ and solve for x:

$$\begin{aligned} u\dot{x} + upx &= uq \\ \frac{d}{dt}(ux) &= uq \\ u(t)x(t) &= \int u(t)q(t)dt + c \\ x(t) &= \frac{1}{u(t)}\left(\int u(t)q(t)dt + c\right) \end{aligned}$$

We have the general solution:

$$x(t) = \frac{1}{u(t)} \int u(t)q(t)dt + \frac{c}{u(t)}. \tag{4}$$

The function u is called an **integrating factor**.

Example. Heat Diffusion

Let's carry out the method in an explicit example.

About this time of year I start to think about summer. I put my root beer in a cooler, but it still gets warm. Let's model its temperature by an ODE:

$$x(t) = \text{ root beer temperature at time } t.$$

The greater the temperature difference between inside and outside, the faster $x(t)$ changes. The simplest (linear) model of this is:

$$\dot{x}(t) = k(T_{\text{ext}}(t) - x(t)),$$

where $T_{ext}(t)$ is the external temperature. This makes sense: when the outside temperature T_{ext} is greater than the inside temperature $x(t)$, then $\dot{x}(t) > 0$ (assuming $k > 0$).

If time permits, add a table showing

$$T_ext > x ==> x' > 0$$
$$T_ext < x ==?x' < 0$$

and then drawing a graph of x' against $x - T_ext$ and deciding that as long as the difference is small it's well approximated by $k(T_ext - x)$. – HB

We get the linear equation:

$$\dot{x} - kx = kT_{\text{ext}}. \tag{5}$$

This is **Newton's law of cooling**; k could depend upon t and we would still have a linear equation, but let's suppose that we are not watching the process for so long that the insulation of the cooler starts to break down![1]

[1]On the other hand, if you plot $\dot{x}$ against x over a larger range of temperatures x, you'll discover that the graph isn't a straight line forever: the cooler melts or the crystal lattice rearranges and the the cooling properties change. If you include these effects then the equation is no longer linear.

Systems and signals analysis:

- The system is the cooler.
- The input signal is the external temperature $T_{\text{ext}}(t)$.
- The output signal or system response is $x(t)$, the temperature in the cooler.

Note that the right-hand side of equation (**??**) is k times the input signal, not the input signal itself. What constitutes the input and output signals is a matter of the interpretation of the equation, not of the equation itself.

To be specific, let $x(0) = 32$ degrees Farenheit, $k = \frac{1}{3}$ and $T_{\text{ext}} = 60 + 6t$, where t denotes hours after 10AM. (The outside temperature is rising linearly.) We get the following differential equation and initial value:

$$\dot{x} + \frac{1}{3}x = 20 + 2t, \qquad x(0) = 32. \tag{6}$$

Solution. We could just plug in to (**??**), but I never do. Instead I apply the method of integrating factors to the differential equation I'm given.

Multiply both sides by u:

$$u\dot{x} + \frac{1}{3}ux = u(20 + 2t). \tag{7}$$

Next, set the left hand side equal to $\frac{d}{dt}(ux)$ and find the integrating factor u:

$$\begin{aligned} u\dot{x} + \frac{1}{3}ux &= u\dot{x} + \dot{u}x \\ \dot{u} &= \frac{1}{3}u \\ u(t) &= e^{\frac{1}{3}t}. \end{aligned}$$

Since any nonzero solution will do, we may choose to let $C = 1$ in the general solution $u(t) = Ce^{\frac{1}{3}t}$. *Deleted sentence about exponential growth equation. – HB*

Now replace the right hand side of (**??**) by $\frac{d}{dt}(ux)$ and solve for x. We chose u so that $\frac{d}{dt}(ux) = u\dot{x} + \frac{1}{3}ux$, so:

$$\begin{aligned} u(\dot{x} + \frac{1}{3}x) &= u(20 + 20t) \\ \frac{d}{dt}(e^{\frac{1}{3}t}x) &= e^{\frac{1}{3}t}(20 + 2t) \\ e^{\frac{1}{3}t}x &= \int e^{\frac{1}{3}t}(20 + 2t)\, dt \\ &= 60e^{\frac{1}{3}t} + 6te^{\frac{1}{3}t} - 18e^{\frac{1}{3}t} + c \qquad \text{(integration by parts)} \\ x(t) &= 60 + 6t - 18 + ce^{-\frac{1}{3}}t \\ &= 42 + 6t + ce^{-\frac{1}{3}t}. \end{aligned}$$

This is the general solution to (**??**). If we let $c = 0$ we get a particular solution which is a polynomial (we'll see later that this is quite easy to determine by other methods). All that remains is to find the value of c that describes the particular behavior of my cooler.

We plug in $t = 0$ and use the initial condition to find c:

$$x(0) = 42 + c \quad \Rightarrow \quad c = -10.$$

The equation describing the temperature inside my cooler is:

$$x(t) = 42 + 6t - 10e^{-\frac{1}{3}t}.$$

4 The Integrating Factor and x_h

Comparing the formula for the integrating factor $u = e^{\int p(t)dt}$ to the solution $x_h = e^{-\int p(t)dt}$ to the homogenous equation (**??**), we get the following expression for x_h:

$$x_h(t) = \frac{1}{u(t)}.$$

The significance of x_h is (partially) described in the next section. *Double check previous sentence. – HB*

5 General = Particular + Homogeneous

I guessed which equation to refer to here. – HB

Note the structure of the general solution (**??**):

$$x = x_p + cu^{-1},$$

where x_p is a solution, *any solution* of (**??**). It's called a **particular solution**, but this is a very poor name because there is nothing particular about it. In this example we chose one with a pretty simple formula:

$$x_p = 42 + 6t.$$

Since $x_h = \frac{1}{u(t)}$, we can rewrite the general solution as $x = x_p + cx_h$.

Very often x_h approaches zero with time, as happens here. It is then called a **transient**. All solutions come to look more and more alike as time goes on. This is a funnel!