Complex Arithmetic

1 History

Most people think that complex numbers arose from attempts to solve quadratic equations, but actually they first appeared in connection with cubic equations. Everyone knew that certain quadratic equations, like

$$x^2 + 1 = 0 \quad \text{or} \quad x^2 + 2x + 5 = 0$$

had no solutions. The problem was with certain cubic equations, for example

$$x^3 - 6x + 2 = 0.$$

This equation was known to have three real roots, given by simple combinations of the expressions

$$A = \sqrt[3]{-1 + \sqrt{-7}} \qquad B = \sqrt[3]{-1 - \sqrt{-7}}. \tag{1}$$

For instance, one of the roots is $A + B$; it may not look like a real number, but it turns out to be one.

What was to be made of the expressions A and B? They were viewed as some sort of "imaginary numbers" which had no meaning in themselves, but which were useful as intermediate steps in calculations which would ultimately lead to the real numbers you were looking for (such as $A + B$).

This point of view persisted for several hundred years. But as more and more applications for these "imaginary numbers" were found, they gradually began to be accepted as valid "numbers" in their own rights, even though they did not measure the length of any line segment. Nowadays we are fairly generous in our use of the word "number": numbers of one sort or another don't have to measure anything, but to merit the name they must belong to a system in which some type of addition, subtraction, multiplication, and division is possible, and where these operations obey those laws of arithmetic one learns in elementary school and has usually forgotten by high school — the commutative, associative, and distributive laws.

2 Definitions

To describe the complex numbers, we use a formal symbol i representing $\sqrt{-1}$; then a **complex number** is an expression of the form:

$$a + ib \qquad a, b \text{ real numbers.} \tag{2}$$

If $a = 0$ or $b = 0$, they are omitted (unless both are 0); thus we write

$$a + i0 = a, \qquad 0 + ib = ib, \qquad 0 + i0 = 0$$

The definition of *equality* between two complex numbers is

$$a + ib = c + id \quad \Leftrightarrow \quad a = c;\ b = d. \tag{3}$$

This shows that the numbers a and b are uniquely determined once the complex number $a + ib$ is given; we call them respectively the **real** and **imaginary** parts of $a + ib$. (It would be more logical to call ib the imaginary part, but this would be less convenient.) In symbols,

$$a = \text{Re}(a + ib) \qquad b = \text{Im}(a + ib) \tag{4}$$

Addition and multiplication of complex numbers are defined in the familiar way, making use of the fact that $i^2 = -1$:

Addition

$$(a + ib) + (c + id) = (a + c) + i(b + d) \tag{5}$$

Multiplication

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc). \tag{6}$$

Division is a little more complicated; what is important is not so much the final formula as the procedure that produces it; assuming $c + id \neq 0$, it is:

Division

$$\frac{a+ib}{c+id} = \frac{a+ib}{c+id} \cdot \frac{c-id}{c-id} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}. \tag{7}$$

This division prodcedure made use of *complex conjugation*: if $z = a + ib$, we define the **complex conjugate** of z to be the complex number

$$\bar{z} = a - ib \qquad \text{(note that } \bar{z}z = a^2 + b^2\text{)}. \tag{8}$$

The size of a complex number is measured by its **absolute value**, or *modulus*, defined by:

$$|z| = |a+ib| = \sqrt{a^2+b^2}; \qquad \text{(thus:} \quad z\bar{z} = |z|^2\text{)}. \tag{9}$$

3 Remarks

One can legitimately object to defining complex numbers simply as formal expressions $a + ib$, on the grounds that "formal expression" is too vague a concept: even if people can handle it, computers cannot. For the latter's sake, we therefore define a complex number to be simply an ordered pair (a, b) of real numbers. With this definition, the arithmetic laws are then defined in terms of ordered pairs; in particular, multiplication is defined by

$$(a,b)(c,d) = (ac - bd, bc + ad).$$

The disadvantage of this approach is that this definition of multiplicatin seems to make little sense. This doesn't bother computers, who do what they are told, but people do better at multiplication by being told to calculate as usual using the relation $i^2 = -1$ to get rid of all the higher powers of i whenever they occur.

Of course, even if you start with the definition using ordered pairs, you can still introduce the special symbol i to represent the ordered pair $(0, 1)$, agree to the abbreviation $(a, 0) = a$, and thus write

$$(a,b) = (a,0) + (0,1)(b,0) = a + ib.$$