Polar Representation

1 The Complex Plane

Complex numbers are represented geometrically by points in the plane: the number $a + ib$ is represented by the point (a, b) in Cartesian coordinates. When the points of the plane are thought of as representing complex numbers in this way, the plane is called the **complex plane**.

By switching to polar coordinates, we can write any non-zero complex number in an alternative form. Letting as usual

$$x = r\cos\theta, \quad y = r\sin\theta$$

we get the **polar form** for a non-zero complex number: assuming $x + iy \neq 0$,

$$x + iy = r(\cos\theta + i\sin\theta). \tag{1}$$

When the complex number is written in polar form,

$$r = |x + iy| = \sqrt{x^2 + y^2}. \qquad \textit{(absolute value, modulus)}$$

We call θ the *polar angle* or the *argument* of $x + iy$. In symbols, one sometimes sees:

$$\theta = \arg(x + iy). \qquad \textit{(polar angle, argument).}$$

The absolute value is uniquely determined by $x + iy$ but the polar angle is not, since it can be increased by any integer multiple of 2π. (The complex number 0 has no polar angle.) To make θ unique, one can specify

$$0 \leq \theta < 2\pi. \qquad \textit{(principal value)}$$

This so-called principal value of the angle is sometimes indicated by writing $\mathrm{Arg}(x + iy)$. For example,

$$\mathrm{Arg}(-1) = \pi, \qquad \arg(-1) = \pm\pi, \pm 3\pi, \pm 5\pi, \cdots$$

Changing between Cartesian and polar representation of a complex number is essentially the same as changing between Cartesian and polar coordinates: the same equations are used.

Example 1. Give the polar form for: $-i, 1+i, 1-i, -1+i\sqrt{3}$.

Solution.

$$
\begin{aligned}
-i &= i\sin\frac{3\pi}{2} & 1+i &= \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) \\
-1+i\sqrt{3} &= 2(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}) & 1-i &= \sqrt{2}(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}).
\end{aligned}
$$

2 Euler's Formula

The abbreviation $\operatorname{cis}\theta$ is sometimes used for $\cos\theta + i\sin\theta$; for students of science and engineering, however, it is important to get used to the exponential form for this expression:

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \textbf{Euler's formula.} \tag{2}$$

Equation (**??**) should be regarded as the *definition* of the exponential of an imaginary power. A good justification for it is found in the infinite series:

$$e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$$

If we substitute $i\theta$ for t in the series and collect the real and imaginary parts of the sum (remembering that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, and so on), we get:

$$
\begin{aligned}
e^{i\theta} &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\
&= \cos\theta + i\sin\theta
\end{aligned}
$$

in view of the infinite series representations for $\cos\theta$ and $\sin\theta$.

Since we only know that the series expansion for e^t is valid when t is a real number, the above argument is only suggestive — it is not a proof of (**??**). What it shows is that Euler's formula (**??**) is formally compatible with the series expansions for the exponential, sine, and cosine functions.

3 Polar Representation

Using the complex exponential, the polar representation (**??**) is written:

$$x + iy = re^{i\theta}. \tag{3}$$

The most important reason for polar representation is that multiplication of complex numbers is particularly simple when they are written in polar form. Indeed, by using Euler's formula (**??**) and the trigonometric addition formulas, it is not hard to show:

$$e^{i\theta}e^{i\theta'} = e^{i(\theta+\theta')}. \tag{4}$$

This gives another justification for the definition (**??**) — it makes the complex exponential follow the same exponential addition rules as the real exponential. The law (**??**) leads to the simple rules for multiplying and dividing complex numbers written in polar form:

multiplication rule

$$r_1e^{i\theta} \cdot r_2e^{i\theta'} = r_1r_2e^{i(\theta+\theta')}. \tag{5}$$

To multiply two complex numbers, you multiply the absolute values and add the angles

reciprocal rule

$$\frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}; \tag{6}$$

division rule

$$\frac{re^{i\theta}}{r'e^{i\theta'}} = \frac{r}{r'}e^{i(\theta-\theta')}. \tag{7}$$

To divide by a complex number, divide by its absolute value and subtract its angle.

The reciprocal rule (**??**) follows from (**??**), which shows that

$$\frac{1}{r}e^{-i\theta} \cdot re^{i\theta} = 1.$$

Using (**??**), we can raise $x + iy$ to a positive integer power by first using $x + iy = re^{i\theta}$; the special case when $r = 1$ is called *DeMoivre's formula*:

$$(x + iy)^n = r^n e^{-n\theta}; \tag{8}$$

DeMoivre's formula

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \tag{9}$$

Example 2. Express:

a) $(1+i)^6$ in Cartesian form;

b) $\dfrac{1+i\sqrt{3}}{\sqrt{3}+i}$ in polar form.

Solution.

a) Change to polar form, use (**??**), then change back to Cartesian form:

$$(1+i)^6 = (\sqrt{2}e^{i\pi/4})^6 = (\sqrt{2})^6 e^{i6\pi/4} = 8e^{i3\pi/2} = -8i.$$

b) Changing to polar form, $\dfrac{1+i\sqrt{3}}{\sqrt{3}+i} = \dfrac{2e^{i\pi/3}}{2e^{i\pi/6}} = e^{i\pi/6}$, using the division rule (**??**).

You can check the answer to (a) by applying the binomial theorem to $(1+i)^6$ and collecting the real and imaginary parts; to (b) by doing the division in the Cartesian form then converting the answer to polar form.

The next section is no longer in Mattuck's notes? I believe there is a mathlet that illustrates this topic. – HB

3.1 Combining pure oscillations of the same frequency.

The equation which does this is widely used in physics and engineering; it can be expressed using complex numbers:

$$A\cos\lambda t + B\sin\lambda t = C\cos(\lambda t + \phi), \qquad \text{where } A + Bi = Ce^{i\phi}; \tag{10}$$

in other words, $C = \sqrt{A^2 + B^2}$, $\phi = \tan^{-1} B/A$. To prove (**??**), we have:

$$\begin{aligned} A\cos\lambda t + B\sin\lambda t &= \mathrm{Re}\left((A+Bi)\cdot(\cos\lambda t + i\sin\lambda t)\right) \\ &= \mathrm{Re}(Ce^{i\phi}\cdot e^{i\lambda t}) \\ &= \mathrm{Re}(Ce^{\lambda t+\phi}) = C\cos(\lambda t + \phi). \end{aligned}$$