Complex Exponentials

Because of the importance of complex exponentials in differential equations, and in science and engineering generally, we go a little further with them. Euler's formula defines the exponential to a pure imaginary power. The definition of an exponential to an arbitrary complex power is:

$$e^{a+ib} = e^a e^{ib} = e^a(\cos b + i \sin b). \tag{1}$$

We stress that the equation (1) is a definition, not a self-evident truth, since up to now no meaning has been assigned to the left-hand side. From (1) we see that

$$\operatorname{Re}(e^{a+ib}) = e^a \cos b, \qquad \operatorname{Im}(e^{a+ib}) = e^a \sin b. \tag{2}$$

The complex exponential obeys the usual law of exponents:

$$e^{z+z'} = e^z e^{z'}, \tag{3}$$

as is easily seen by combining (1) with the multiplication rule for complex numbers.

The complex exponential is expressed in terms of the sine and cosine by Euler's formula. Conversely, the sin and cos functions can be expressed in terms of complex exponentials. There are two important ways of doing this, both of which you should learn:

$$\cos x = \operatorname{Re}(e^{ix}), \qquad \sin x = \operatorname{Im}(e^{ix}); \tag{4}$$

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}), \qquad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix}). \tag{5}$$

The equations in (5) follow easily from Euler's formula; their derivation is left for the exercises. Here are some examples of their use. *Do our exercises include this derivation? If not, change to "...is left as an exercise". – HB*

Example. Express $\cos^3 x$ in terms of the functions $\cos nx$, for suitable n.

Solution. We use (5) and the binomial theorem, then (5) again:

$$\begin{aligned} \cos^3 x &= \frac{1}{8}(e^{ix} + e^{-ix})^3 \\ &= \frac{1}{8}(e^{3ix} + 3e^{ix} + 3e^{-ix} + e^{-3ix}) \\ &= \frac{1}{4}\cos 3x + \frac{3}{4}\cos x. \end{aligned}$$

I left out the little square "end of proof" sybmol here and below. – HB

As a preliminary to the next example, we note that a function like

$$e^{ix} = \cos x + i \sin x$$

is a *complex-valued function of the real variable x*. Such a function may be written as

$$u(x) + iv(x) \qquad u, v \text{ real-valued}$$

and its derivative and integral with respect to x are defined to be

$$\text{a) } D(u + iv) = Du + iDv \qquad \text{b) } \int (u + iv)dx = \int udx + i \int vdx. \qquad (6)$$

From this it follows by a calculation that

$$D(e^{(a+ib)x}) = (a + ib)e^{(a+ib)x},$$

and therefore

$$\int e^{(a+ib)x}dx = \frac{1}{a + ib}e^{(a+ib)x}. \qquad (7)$$

Example. Calculate $\int e^x \cos 2x \, dx$ by using complex exponentials.

Solution. The usual method is a tricky use of two successive integration by parts. Using complex exponentials instead, the calculation is straightforward. We have

$$e^x \cos 2x = \text{Re}(e^{(1+2i)x}), \qquad \text{by (1) or (2); therefore}$$
$$\int e^x \cos 2x \, dx = \text{Re}(\int e^{(1+2i)x}dx), \text{ by (6)b.}$$

Calculating the integral,

$$\int e^{(1+2i)x}dx = \frac{1}{1 + 2i}e^{(1+2i)x} \qquad \text{by (7)}$$
$$= \left(\frac{1}{5} - \frac{2}{5}i\right)(e^x \cos 2x + ie^x \sin 2x),$$

using (1) and complex division. According to the second line above, we want the real part of this last expression. Multiply and take the real part; you get

$$\frac{1}{5}e^x \cos 2x + \frac{2}{5}e^x \sin 2x.$$

In this differential equations course, we will make free use of complex exponentials in solving differential equations, and in doing formal calculations like the ones above. This is standard practice in science and engineering, and you need to get used to it. *I'm worried that Mattuck's phrases like "and you need to get used to it" are going to clash badly with Haynes' announcements like "the sum of two sinusoids is another sinusoid!". I shall deal with this concern by pointing out phrases like the one here. – HB*