Finding n-th Roots

To solve linear differential equations with constant coefficients, we need to be able to find the real and complex roots of polynomial equations. Though a lot of this is done today with calculators and computers, one still has to know how to do an important special case by hand: finding the roots of

$$z^n = \alpha,$$

where α is a complex number, i.e., finding the n-th roots of α. Polar representation will be a big help in this.

Let's begin with a special case: the ***n*-th roots of unity**: the solutions to

$$z^n = 1.$$

To solve this equation, we use polar representation for both sides, setting $z = re^{i\theta}$ on the left, and using all possible polar angles on the right; using the exponential law to multiply, the above equation then becomes

$$r^n e^{in\theta} = 1 \cdot e^{(2k\pi i)}, \qquad k = 0, \pm 1, \pm 2, \cdots .$$

Equating the absolute values and the polar angles of the two sides gives

$$r^n = 1, \qquad n\theta = 2k\pi, \qquad k = 0, \pm 1, \pm 2, \cdots ,$$

from which we conclude that

$$r = 1, \qquad\qquad \theta = \frac{2k\pi}{n}, \qquad k = 0, 1, \cdots, n-1. \tag{1}$$

In the above, we get only the value $r = 1$, since r must be real and non-negative. We don't need any integer values of k other than $0, \cdots, n-1$, since they would not produce a complex number different from the above n numbers. That is, if we add an, an integer multiple of n, to k, we get the same complex number:

$$\theta' = \frac{2(k+an)\pi}{n} = \theta + 2a\pi; \qquad \text{and} \qquad e^{i\theta'} = e^{i\theta}, \qquad \text{since } e^{2a\pi i} = (e^{2\pi i})^a = 1.$$

We conclude from (1) therefore that

$$\textit{the n-th roots of 1 are the numbers } \; e^{2k\pi i/n},\ k = 0, \cdots, n-1. \tag{2}$$

This shows there are n complex n-th roots of unity. They all lie on the unit circle in the complex plane, since they have absolute value 1; they are evenly spaced around the unit circle, starting with 1; the angle between two consecutive ones is $2\pi/n$. These facts are illustrated in Figure 1 for the case $n = 6$.

Figure 1: The six solutions to the equation $z^6 = 1$ lie on a unit circle in the complex plane.

From (2), we get another notation for the roots of unity (ζ is the Greek letter "zeta"):

$$\textit{the } n\textit{-th roots of 1 are } 1, \zeta, \zeta^2, \cdots, \zeta^{n-1}, \text{ where } \zeta = e^{2\pi i/n}. \tag{3}$$

We now generalize the above to find the n-th roots of an arbitrary complex number w. We begin by writing w in polar form:

$$w = re^{i\theta}; \qquad \theta = \text{Arg}w,\ 0 \leq \theta < 2\pi,$$

i.e., θ is the principal value of the polar angle of w. Then the same reasoning as we used above shows that if z is an n-th root of w, then

$$z^n = w = re^{i\theta} \quad \text{so} \quad z = \sqrt[n]{r} e^{i(\theta + 2k\pi)/n}, \quad k = 0, 1, \cdots, n-1. \tag{4}$$

Comparing this with (3), we see that these n roots can be written in the suggestive form

$$\sqrt[n]{w} = z_0,\ z_0\zeta,\ z_0\zeta^2, \cdots, z_0\zeta^{n-1}, \qquad \text{where } z_0 = \sqrt[n]{r} e^{i\theta/n}. \tag{5}$$

As a check, we see that all of the n complex numbers in (5) satisfy $z^n = w$:

$$\begin{aligned} (z_0\zeta^i)^n &= z_0^n \zeta^{ni} = z_0^n \cdot 1^i, && \text{since } \zeta^n = 1, \text{ by (3);} \\ &= w, && \text{by the definition (5) of } z_0 \text{ and (4).} \end{aligned}$$

Example. Find in Cartesian form all values of a) $\sqrt[3]{1}$ b) $\sqrt[4]{1}$

Solution. a) According to (3), the cube roots of 1 are $1, \omega$, and ω^2, where

$$\omega = e^{2\pi i/3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$
$$\omega^2 = e^{-2\pi i/3} = \cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

The greek letter ω ("omega") is traditionally used for this cube root. Note that for the polar angle of ω^2 we used $-2\pi/3$ rather than the equivalent angle $4\pi/3$, in order to take advantage of the identities

$$\cos(-x) = \cos x \qquad \sin(-x) = -\sin x.$$

Note that $\omega^2 = \overline{\omega}$. Another way to do this problem would be to draw the position of ω^2 and ω on the unit circle and use geometry to figure out their coordinates.

b) To find $\sqrt[4]{i}$, we can use (5). We know that $\sqrt[4]{1} = 1, i, -1, -i$ (either by drawing the unit circle picture or by using (3)). Therefore by (5), we get

$$\begin{aligned} \sqrt[4]{i} &= z_0, z_0 i, -z_0, -z_0 i, && \text{where } z_0 = e^{\pi i/8} = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8}; \\ &= a + ib, -b + ia, -a - ib, b - ia && \text{where } z_0 = a + ib = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8}. \end{aligned}$$

Example. Solve the equation $x^6 - 2x^3 + 2 = 0$.

Solution. Treating this as a quadratic equation in x^3, we solve the quadratic by using the quadratic formula; the two roots are $1 + i$ and $1 - i$ (check this!), so the roots of the original equation satisfy either

$$x^3 = 1 + i \quad \text{or} \quad x^3 = 1 - i.$$

This reduces the problem to finding the cube roots of the two complex numbers $1 \pm i$. We begin by writing them in polar form:

$$1 + i = \sqrt{2}e^{\pi i/4}, \qquad 1 - i = \sqrt{2}e^{-\pi i/4}.$$

(Once again, note the use of the negative polar angle for $1 - i$, which is more convenient for calculations.) The three cube roots of the first of these are (by (4)),

$$
\begin{aligned}
\sqrt[6]{2}e^{\pi i/12} &= \sqrt[6]{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) \\
\sqrt[6]{2}e^{3\pi i/4} &= \sqrt[6]{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right), \quad \text{since } \frac{\pi}{12} + \frac{2\pi}{3} = \frac{3\pi}{4}; \\
\sqrt[6]{2}e^{-7\pi i/12} &= \sqrt[6]{2}\left(\cos\frac{7\pi}{12} - i\sin\frac{7\pi}{12}\right), \quad \text{since } \frac{\pi}{12} - \frac{2\pi}{3} = -\frac{7\pi}{12}.
\end{aligned}
$$

The second cube root can also be written as $\sqrt[6]{2}\left(\frac{-1+i}{\sqrt{2}}\right) = \frac{-1+i}{\sqrt[3]{2}}$.

This gives three of the cube roots. The other three are the cube roots of $1 - i$, which may be found by replacing i by $-i$ everywhere (i.e., taking the complex conjugate).

The cube roots can also be described according to (5) as

$$
z_1,\ z_1\omega,\ z_1\omega^2 \ \text{ and } \ z_2,\ z_2\omega,\ z_2\omega^2 \ \text{ where } \ z_1 = \sqrt[6]{2}e^{\pi i/12},\ z_2 = \sqrt[6]{2}e^{-\pi i/12}.
$$

Should this have a concluding paragraph? What about subsections? – HB