The Sinusoidal Identity

The sum of two sinusoidal functions of the same frequency is another sinusoidal function with that frequency! For any real constants a and b,

$$a \cos \omega t + b \sin \omega t = A \cos(\omega t - \phi) \tag{1}$$

where A and ϕ can be described in at least three ways:

$$A = |a\mathbf{i} + b\mathbf{j}| \text{ and } \phi \text{ is the angle rotating } \mathbf{i} \text{ into } \operatorname{dir}(a\mathbf{i} + b\mathbf{j}); \tag{2}$$

$$A = \sqrt{a^2 + b^2}, \quad \phi = \tan^{-1} \frac{b}{a}; \tag{3}$$

$$a + bi = Ae^{i\phi}. \tag{4}$$

Conversely, we have

$$a = A \cos \phi \text{ and } b = A \sin \phi. \tag{5}$$

Geometrically this is summarized by the triangle in Figure 1:

Figure 1: $a + bi = Ae^{i\phi}$.

One proof of (1) is a simple application of the cosine addition formula

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta.$$

We will give an equivalent proof using Euler's formula and complex arithmetic.

The triangle in Figure 1 is the standard polar coordinates triangle. It shows $a + ib = Ae^{i\phi}$ or $a - ib = Ae^{-i\phi}$. Thus

$$\begin{aligned}
A \cos(\omega t - \phi) &= \operatorname{Re}(Ae^{i(\omega t - \phi)}) \\
&= \operatorname{Re}(e^{i\omega t} \cdot Ae^{-i\phi}) \\
&= \operatorname{Re}((\cos \omega t + i \sin \omega t) \cdot (a - ib)) \\
&= \operatorname{Re}(a \cos \omega t + b \sin \omega t + i(a \sin \omega t - b \cos \omega t)) \\
&= a \cos \omega t + b \sin \omega t.
\end{aligned}$$

We should stress the importance of the trigonometric identity (1). It shows that *any* linear combination of $\cos(\omega t)$ and $\sin(\omega t)$ is not only periodic of period $\frac{2\pi}{\omega}$, but is also sinusoidal. If you try to add $\cos(\omega t)$ to $\sin(\omega t)$ "by hand", you will probably agree that this is not at all obvious.

We will call $A\cos(\omega t - \phi)$ **amplitude-phase form** and $a\cos\omega t + b\sin\omega t$ **rectangular** or **Cartesian form**. You should be familiar amplitude-phase form; we usually prefer it because both amplitude and phase have geometric and physical meaning for us. *I changed "geometrical" to "geometric". Perhaps both are correct, but I find I have a strong preference. – HB*