**Exercises on the four fundamental subspaces**

**Problem 10.1:** (3.6 #11. *Introduction to Linear Algebra:* Strang) $A$ is an $m$ by $n$ matrix of rank $r$. Suppose there are right sides $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has *no solution*.

a) What are all the inequalities ($<$ or $\leq$) that must be true between $m, n$, and $r$?

b) How do you know that $A^T\mathbf{y} = \mathbf{0}$ has solutions other than $\mathbf{y} = \mathbf{0}$?

**Solution:**

a) The rank of a matrix is always less than or equal to the number of rows and columns, so $r \leq m$ and $r \leq n$. The second statement tells us that the column space is not all of $\mathbb{R}^n$, so $r < m$.

b) These solutions make up the left nullspace, which has dimension $m - r > 0$ (that is, there are nonzero vectors in it).

**Problem 10.2:** (3.6 #24.) $A^T\mathbf{y} = \mathbf{d}$ is solvable when $\mathbf{d}$ is in which of the four subspaces? The solution $\mathbf{y}$ is unique when the __________ contains only the zero vector.

**Solution:** It is solvable when $\mathbf{d}$ is in the row space, which consists of all vectors $A^T\mathbf{y}$. The solution $\mathbf{y}$ is unique when the **left nullspace** contains only the zero vector.