Exercises on projections onto subspaces

Problem 15.1: (4.2 #13. *Introduction to Linear Algebra:* Strang) Suppose A is the four by four identity matrix with its last column removed; A is four by three. Project $\mathbf{b} = (1, 2, 3, 4)$ onto the column space of A. What shape is the projection matrix P and what is P?

Solution: P will be four by four since we are projecting a 4-dimensional vector to another 4-dimensional vector. We will have:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This can be seen by observing that the column space of A is the $wxy-$space, so we just need to subtract the z coordinate from the 4-dimensional vector (w, x, y, z) we're projecting. The projection of $\mathbf{b}$ is therefore:

$$\mathbf{p} = P\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}.$$

Problem 15.2: (4.2 #17.) If $P^2 = P$, show that $(I - P)^2 = I - P$. For the matrices A and P from the previous question, P projects onto the column space of A and $I - P$ projects onto the ________.

Solution:

$$(I - P)^2 = I^2 - IP - PI + P^2 = I - 2P + P^2 = I - 2P + P = I - P.$$

Using the matrices A and P from the previous question,

$$I - P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

projects onto the **left nullspace** of A.