Exercises on properties of determinants

Problem 18.1: (5.1 #10. *Introduction to Linear Algebra:* Strang) If the entries in every row of a square matrix A add to zero, solve $A\mathbf{x} = \mathbf{0}$ to prove that $\det A = 0$. If those entries add to one, show that $\det(A - I) = 0$. Does this mean that $\det A = 1$?

Solution: If the entries of every row of A sum to zero, then $A\mathbf{x} = 0$ when $\mathbf{x} = (1, \ldots, 1)$ since each component of $A\mathbf{x}$ is the sum of the entries in a row of A. Since A has a non-zero nullspace, it is not invertible and $\det A = 0$.

If the entries of every row of A sum to one, then the entries in every row of $A - I$ sum to zero. Hence $A - I$ has a non-zero nullspace and $\det(A - I) = 0$.

If $\det(A - I) = 0$ it is **not** necessarily true that $\det A = 1$. For example, the rows of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ sum to one but $\det A = -1$.

Problem 18.2: (5.1 #18.) Use row operations and the properties of the determinant to calculate the three by three "Vandermonde determinant":

$$\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (b-a)(c-a)(c-b).$$

Solution: Using row operations and properties of the determinant, we have:

$$
\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = \det \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 1 & c & c^2 \end{bmatrix}
$$

$$
= \det \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix}
$$

$$
= (b-a) \det \begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 1 & c-a & c^2-a^2 \end{bmatrix}
$$

$$
= (b-a) \det \begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & (c-a)(c-b) \end{bmatrix}
$$

$$
= (b-a)(c-a)(c-b) \det \begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
= (b-a)(c-a)(c-b) \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
= (b-a)(c-a)(c-b). \checkmark
$$