

## Exercises on elimination with matrices

**Problem 2.1:** In the two-by-two system of linear equations below, what multiple of the first equation should be subtracted from the second equation when using the method of elimination? Convert this system of equations to matrix form, apply elimination (what are the pivots?), and use back substitution to find a solution. Try to check your work before looking up the answer.

$$\begin{aligned}2x + 3y &= 5 \\6x + 15y &= 12\end{aligned}$$

**Solution:** One subtracts 3 times the first equation from the second equation in order to eliminate the  $6x$ .

To convert to matrix form, use the general format  $A\mathbf{x} = \mathbf{b}$ :

$$\begin{aligned}2x + 3y &= 5 \\6x + 15y &= 12\end{aligned} \longrightarrow \begin{bmatrix} 2 & 3 \\ 6 & 15 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}.$$

We then apply elimination on matrix  $A$ . Using the first pivot (the number 2 in the upper left corner of  $A$ ), we subtract three times the first row from the second row to get:

$$A = \begin{bmatrix} 2 & 3 \\ 6 & 15 \end{bmatrix} \longrightarrow U = \begin{bmatrix} 2 & 3 \\ 0 & 6 \end{bmatrix}$$

where  $U$  is an upper triangular matrix with pivots 2 and 6. Doing the same to the right side  $\mathbf{b} = (5, 12)$  gives a new equation of the form  $U\mathbf{x} = \mathbf{c}$ :

$$\begin{bmatrix} 2 & 3 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}.$$

To solve our new equation, we use back substitution:

$$6y = -3 \longrightarrow \boxed{y = -\frac{1}{2}}$$

and

$$2x + 3y = 5 \longrightarrow 2x + 3\left(-\frac{1}{2}\right) = 5 \longrightarrow 2x = 5 + \frac{3}{2} = \frac{13}{2} \longrightarrow \boxed{x = \frac{13}{4}}$$

We know that our solution fulfills the first equation; let's make sure that our values fulfill the second equation as a check on our work:

$$6x + 15y = 6\left(\frac{13}{4}\right) + 15\left(-\frac{1}{2}\right) = \frac{78 - 30}{4} = 12 \checkmark$$

**Problem 2.2:** (2.3 #29. *Introduction to Linear Algebra: Strang*) Find the triangular matrix  $E$  that reduces "Pascal's matrix" to a smaller Pascal:

$$E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}.$$

Which matrix  $M$  (multiplying several  $E$ 's) reduces Pascal all the way to  $I$ ?

**Solution:**

$$\text{The matrix is } E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

One can eliminate the second column with the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

and the third column with the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Multiplying these together, we get

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}.$$

Since  $M$  reduces the Pascal matrix to  $I$ ,  $M$  must be the inverse matrix!