Exercises on Cramer's rule, inverse matrix, and volume

Problem 20.1: (5.3 #8. *Introduction to Linear Algebra:* Strang) Suppose

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}.$$

Find its cofactor matrix C and multiply AC^T to find $\det(A)$.

$$C = \begin{bmatrix} 6 & -3 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \text{ and } AC^T = \underline{\qquad}.$$

If you change $a_{1,3} = 4$ to 100, why is $\det(A)$ unchanged?

Solution: We fill in the cofactor matrix C and then multiply to obtain AC^T:

$$C = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix}$$

and

$$AC^T = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 6 & 3 & -6 \\ -3 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 3I.$$

Since $AC^T = \det(A)I$, we have $\det(A) = 3$. If 4 is changed to 100, $\det(A)$ is unchanged because the cofactor of that entry is 0, and thus its value does not contribute to the determinant.

Problem 20.2: (5.3 #28.) Spherical coordinates ρ, ϕ, θ satisfy

$$x = \rho \sin\phi \cos\theta, \ y = \rho \sin\phi \sin\theta \text{ and } z = \rho \cos\phi.$$

Find the three by three matrix of partial derivatives:

$$\begin{bmatrix} \partial x/\partial\rho & \partial x/\partial\phi & \partial x/\partial\theta \\ \partial y/\partial\rho & \partial y/\partial\phi & \partial y/\partial\theta \\ \partial z/\partial\rho & \partial z/\partial\phi & \partial z/\partial\theta \end{bmatrix}.$$

Simplify its determinant to $J = \rho^2 \sin\phi$. In spherical coordinates,

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

is the volume of an infinitesimal "coordinate box."

Solution: The rows are formed by the partials of x, y, and z with respect to ρ, ϕ, and θ:

$$\begin{bmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{bmatrix}.$$

Expanding its determinant J along the bottom row, we get:

$$\begin{aligned} J &= \cos\phi \begin{bmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{bmatrix} \\ &\quad -(-\rho\sin\phi) \begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{bmatrix} + 0 \\ &= \cos\phi(\rho^2\cos\phi\sin\phi\cos^2\theta + \rho^2\cos\phi\sin\phi\sin^2\theta) \\ &\quad +\rho\sin\phi(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta) \\ &= \cos\phi(\rho^2\cos\phi\sin\phi(\cos^2\theta + \sin^2\theta)) + \rho\sin\phi(\rho\sin^2\phi(\cos^2\theta + \sin^2\theta)) \\ &= \cos\phi(\rho^2\cos\phi\sin\phi) + \rho^2\sin^3\phi \\ &= \rho^2\sin\phi(\cos^2\phi + \sin^2\phi) \\ J &= \rho^2\sin\phi. \end{aligned}$$