Exercises on positive definite matrices and minima

Problem 27.1: (6.5 #33. *Introduction to Linear Algebra:* Strang) When A and B are symmetric positive definite, AB might not even be symmetric, but its eigenvalues are still positive. Start from $AB\mathbf{x} = \lambda\mathbf{x}$ and take dot products with $B\mathbf{x}$. Then prove $\lambda > 0$.

Solution:

$$\begin{aligned} AB\mathbf{x} &= \lambda\mathbf{x} \\ (AB\mathbf{x})^T B\mathbf{x} &= (\lambda\mathbf{x})^T B\mathbf{x} \\ (B\mathbf{x})^T A^T B\mathbf{x} &= \lambda\mathbf{x}^T B\mathbf{x} \\ (B\mathbf{x})^T A(B\mathbf{x}) &= \lambda(\mathbf{x}^T B\mathbf{x}). \end{aligned}$$

where $A^T = A$ because A is symmetric. Since A is positive definite we know $(B\mathbf{x})^T A(B\mathbf{x}) > 0$, and since B is positive definite $\mathbf{x}^T B\mathbf{x} > 0$. Hence, λ must be positive as well.

Problem 27.2: Find the quadratic form associated with the matrix $\begin{bmatrix} 1 & 5 \\ 7 & 9 \end{bmatrix}$. Is this function $f(x,y)$ always positive, always negative, or sometimes positive and sometimes negative?

Solution: To find the quadratic form, compute $\mathbf{x^T}A\mathbf{x}$:

$$\begin{aligned} f(x,y) &= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 7 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= x(x+5y) + y(7x+9y) \\ &= \mathbf{x^2 + 12xy + 9y^2}. \end{aligned}$$

This expression can be positive, e.g. when $y = 0$ and $x \neq 0$.

The expression will sometimes be negative because A is not positive definite. For instance, $f(2,-2) = -8$. Thus the quadratic form associated with the matrix A is **sometimes positive and sometimes negative**. Another way to reach this conclusion is to note that $\det A = -26$ is negative and so A is not positive definite.