**Exercises on singular value decomposition**

**Problem 29.1:** (Based on 6.7 #4. *Introduction to Linear Algebra:* Strang) Verify that if we compute the singular value decomposition $A = U\Sigma V^T$ of the Fibonacci matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$,

$$\Sigma = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{\sqrt{5}-1}{2} \end{bmatrix}.$$

**Solution:**

$$A^T A = AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

The eigenvalues of this matrix are the roots of $x^2 - 3x + 1$, which are $\dfrac{3 \pm \sqrt{5}}{2}$. Thus we have:

$$\sigma_1^2 = \frac{3+\sqrt{5}}{2} \quad \text{and} \quad \sigma_2^2 = \frac{3-\sqrt{5}}{2}.$$

To check that $\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$, we will square the entries of the matrix $\Sigma$ given above.

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{3+\sqrt{5}}{2}. \checkmark$$

$$\left(\frac{\sqrt{5}-1}{2}\right)^2 = \frac{5-2\sqrt{5}+1}{4} = \frac{3-\sqrt{5}}{2}. \checkmark$$

**Problem 29.2:** (6.7 #11.) Suppose $A$ has orthogonal columns $\mathbf{w}_1$, $\mathbf{w}_2$, ..., $\mathbf{w}_n$ of lengths $\sigma_1$, $\sigma_2$, ..., $\sigma_n$. Calculate $A^T A$. What are $U$, $\Sigma$, and $V$ in the SVD?

**Solution:** Since the columns of $A$ are orthogonal, $A^T A$ is a diagonal matrix with entries ${\sigma_1}^2$, ..., ${\sigma_n}^2$. Since $A^T A = V\Sigma^2 V^T$, we find that $\Sigma^2$ is the matrix with diagonal entries ${\sigma_1}^2$, ...,${\sigma_n}^2$ and thus that $\Sigma$ is the matrix with diagonal entries $\sigma_1$, ..., $\sigma_n$.

Referring again to the equation $A^T A = V\Sigma^2 V^T$, we conclude also that $V = I$.

The equation $A = U\Sigma V^T$ then tells us that $U$ must be the matrix whose columns are $\frac{1}{\sigma_i}\mathbf{w}_i$.